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Dr. H. C. Christofferson
New President of the National Council of Teachers of Mathematics

THE MATHEMATICS TEACHER

Volume XXXI Number 4

Edited by William David Reeve

Geometry a Way of Thinking*

By H. C. Christofferson
Miami University, Oxford, Ohio

A student of geometry who gets the most out of his work will learn two types of things. First he will learn many facts and principles about form and space. These will enable him to solve numerous problems. Second, he will learn how statements are proved and the nature of sound proof. He will never again be content with statements which merely seem to be true; he will demand proof and will require the proof to be correct, clear, and precise. Because of the simple and concrete nature of the ideas with which geometry deals, the nature of rigorous proof is illustrated with relative ease.

The Indiana World Memorial Shrine in which this meeting is held is a striking and lasting tribute to Geometry. I would indeed be remiss in my appreciation of the contribution of mathematics to civilization, if, seeing these impressive columns, graceful arches, and inspiring symmetries, I did not emphasize that the chief purpose of teaching geometry is to teach geometry. Then, too, we remember that even now we are almost in the shadow of the State House where it is reported that a bill was at one time introduced to change the value of pi to 3, and that it was only through strenuous opposition by the mathematicians of Indiana that the bill failed to become a law. We are impressed by the need for and the value of the facts and principles of geometry in many phases of society. Therefore, the major purpose of geometry teaching should ever be to provide a knowledge of the great basal propositions and to develop the ability to apply those propositions in the solution of problems.

It shall be the purpose of this paper, however, to emphasize a secondary objective. That is, to show how, when, and where the pattern of thinking so clearly illustrated in geometry can be and is used in non-geometric situations. The following are illustrative of good thinking in a geometric and in a non-geometric situation, in a simple geometry problem and in a problem that is non-geometric.

A geometry theorem: If a triangle is isosceles, the angles opposite the equal sides are equal.

The function of proof in this proposition is to establish a cause and effect relation between the equality of the sides and the equality of the angles. The latter must be an inevitable consequence of the former. It is quite evident that the proof consists of first, a definition of isosceles; second, the construction of an angle bisector; third, the congruence of triangles

* A paper read to the National Council of the Teachers of Mathematics at their meeting with the Mathematical Association of America in Indianapolis, December, 1937.

by side-angle-side. As a consequence of this proof, we have established the fact that whenever a triangle exists or is constructed with two sides equal, then it must always follow that the angles opposite those sides are also equal.

A theorem in non-geometric thinking: If all standing water is covered with a film of oil, then there will be no mosquitoes. It is evident that this statement has a premise or hypothesis "all standing water is covered with a film of oil," and a conclusion, "there will be no mosquitoes."

The proof consists of showing that the conclusion necessarily follows from the premise. The following items are suggestive of proof: (1) The mosquito lays its eggs in only standing water. (2) The eggs hatch out into larvae which float on the surface of the water where they must breathe air. (3) Oil being lighter than water covers the surface and prevents their breathing. (4) Larvae cannot live without air. (5) Therefore, as soon as the present mosquitoes die there will be no more, since the young will all be killed.

Each of these statements must be supported by proper and convincing authority, otherwise the chain of reasoning is broken. For instance, if some mosquitoes laid their eggs in the ground or if some larvae got oxygen from the water like fish do, or even if the larvae could float on the oil, or their breathing tubes extend through the oil, or other exceptions made, the whole argument would be void. These statements must all be established. They are like postulates or previous theorems in geometry. If any of them is not true, the entire reasoning structure crumbles to nothing.

Notice too that there are several terms which need definition: *all, standing, film.* Naturally, most terms are undefined, that is, their meaning is known by common usage. But a "film" of oil, how thin is that? Does "standing" water include what may be standing for a day or so after a heavy rain? Does "all" standing water include water in an old bucket back of the garage, a fresh water lake, a swamp, the ocean? Just what does it include?

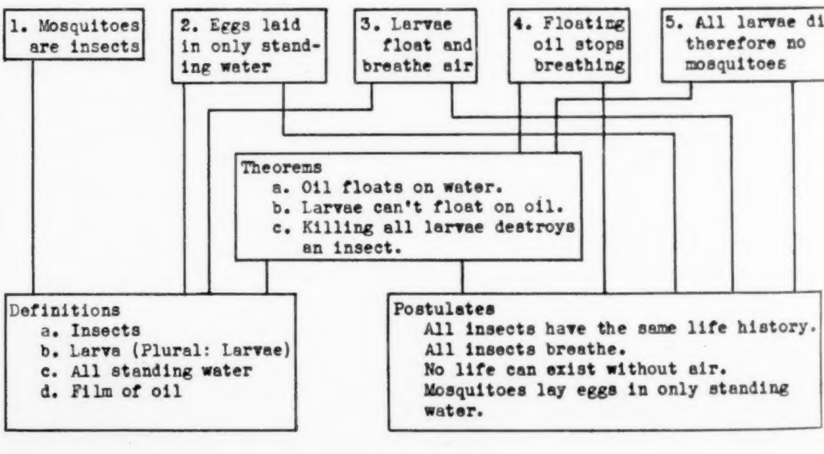
The following chart endeavors to picture the dependence of proof upon definitions, previous theorems, and postulates. A corresponding chart of a proof in geometry could easily be made and is familiar to most readers.

Proposition: If all standing water is covered with a film of oil, there will be no mosquitoes.

Hypothesis or *Premise:* All standing water is covered with a film of oil.

Conclusion: There will be no mosquitoes.

Proof:



Clear, accurate, precise thinking in any situation will involve many undefined terms and some definitions, several statements accepted without proof like postulates and some statements for which proof will be necessary. Inductive reasoning and deductive reasoning are both involved, as are also direct and indirect proof. The following sections of this paper will deal with these phases of good thinking. Some commonly used terms which really need careful definition will be listed. Several systems of postulates in various fields of thought will be suggested. Then, inductive and deductive thinking in direct and indirect proof will be shown and illustrated, briefly.

Definitions

Let us examine a few commonly used terms which we informally define.

Grease the car is a common phrase which seems to have a rather special meaning. It does not include greasing the wheel bearings, it does not include oil for the cylinders nor for the main bearings on the crank shaft, it does not include grease for differential or transmission. Sometimes it seems to include grease on the upholstery and fenders, although this, too, is not supposed to be included. Just what does the phrase mean?

"No U-turn," "no turn," "one-way," "slow school," "slow children," "slow cattle crossing," "stop school bus," "slow men working," etc., are all phrases which will be recognized by any motorist as needing definition. Their real meaning is quite different in many cases from the apparent one.

After one has bought a lot upon which to build a house, the word *deed* acquires a new meaning, and also the words *lot* and *title*.

Bull and *bear* are common words, yet when applied to the stock market one has to know the technical meaning to understand what is said or written.

The following are a few common business terms which might readily be grossly misunderstood unless carefully defined. *Asset* or *liability*, *capital*, *days of grace*, *executor*, *face*, *F.O.B.*, *C.O.D.*, *gross*, *insolvent*, *maturity*, *net*, *par*, *premium*, *sinking fund*, *stock*, *preferred stock*, *teller*, *voucher*, *will*. Some "preferred" stock may not be preferred and some "common" stock may be far from common. "30 days net" is not a hair net that lasts for 30 days.

Strike, *ball*, *foul*, *hit*, *touchdown*, *last quarter*, *love game*, *par*, *birdie*, *spoon*, *trap*, *two spades*, *vulnerable*, *rubber*, are merely samples of words whose meaning may be rather technical. In fact, they may have quite different meanings in different settings. Par in golf does not mean 100. The number of terms needing careful definition and which we use very commonly is large.

Postulates

There are many systems of postulates outside of geometry. "We hold these truths to be self-evident, that all men are created equal, that they are endowed by their creator with certain unalienable rights, that among these are life, liberty and the pursuit of happiness. That to secure these rights governments are instituted among men, deriving their just powers from the consent of the governed, . . . " This quotation from the Declaration of Independence suggests a system of postulates for government. In fact our constitution can readily be considered our fundamental system of governmental postulates.

To realize how important these principles are one needs only to consider what other nations might do with them. Italy, Japan, Germany, Russia all have different sets of postulates which they accept without proof as the basis for their government. All their reasoning about war, conquest, rights of individuals, and all other affairs of government are determined by their postulates of government.

"Strike three, batter out," to a thoughtful student may suggest the question why

three "strikes" and four "balls" rather than two or four strikes. Also why three "outs," nine "innings," and nine, eleven, five or four players on a team. The rules of a game are postulates. They are accepted without proof and then used in the game to prove which team wins points and the game.

"Honesty is the best policy," is accepted by some of us as true. We do not argue the matter. It is a postulate governing our actions. Some men do not accept this postulate and others of a similar nature, and therefore their actions are different.

The Apostles' Creed, "I believe in God the father, the maker of heaven and earth, and in Jesus Christ his only son our Lord, . . . " is in a sense the postulational basis for Christianity. If one accepts these statements as true, he then reasons on the basis of them and his life and actions are governed by them.

Most of the so-called laws of science are just postulates. That is, they are accepted as true in an effort to account for certain facts or happenings. Whenever the time is reached that they do not seem to be consistent with happenings, they are supplanted by new postulates. The law of gravitation, the law of levers, are illustrations of accepted statements which seem to account for certain results and have been used as postulates.

Inductive and Deductive Thinking

Geometry is ideally a deductive science, yet much of the thinking is inductive in nature at first. Finally, conclusions are proved by rigorous deductive reasoning. The difference between inductive and deductive thinking is best shown by an illustration. Suppose a triangle is constructed on a certain base line with two angles each 50°, or as nearly so as humanly possible. If the sides opposite these angles are measured with accurate instruments, they may differ by as much as .001 or possibly even .01 of an inch.

Yet when we prove the theorem that "if two angles of a triangle are equal, the triangle is isosceles," we are certain that the two sides are exactly equal. We proved this by using only previously proved or accepted statements and statements which were completely general.

This is deductive reasoning, the proving of one general statement by showing how it is based on other general statements previously accepted. Such proof does not depend upon the size of the angles or sides in degrees or inches, nor the material of which the triangle is constructed. Neither does the language spoken, nor the climate, nor the form of government influence the result. Deductive reasoning is not handicapped by the limitations of instruments of observation or measure, materials of construction, nor even temperature fluctuations. It is exact, accurate, unquestionable. Such proof, whether in geometry or elsewhere is often spoken of as being mathematically rigorous. The meaning of mathematical rigor is that the conclusion is the correct and only one which can follow from the hypothesis, the postulates, and the other theorems or conclusions upon which it is based.

Much of our so-called thinking is not rigorous, but purely inductive. For instance, Mary has red hair and is jolly, Tom has red hair and is jolly; therefore, all red-haired people are jolly. Bobby Jones uses a certain stance and grip in driving a golf ball; therefore, if I use the same stance and grip I should be able to drive like Jones does. Heifitz practiced ten hours a day; therefore, if James practices on his violin ten hours a day he will be a Heifetz.

Much of modern advertising depends for its efficiency upon our non-rigorous inductive thinking. Many tobacco advertisements feature beautiful girls or beautiful girls escorted by handsome men. Almost unconsciously any girl will generalize: "These beautiful girls smoke X cigarettes; therefore, all beautiful girls should smoke X cigarettes," or "This girl who smokes X cigarettes has a fine boy friend;

therefore, if I smoke X cigarettes, I too will have a fine boy friend."

Pictures of charming girls in abbreviated bathing suits and pictures of beautiful scenery such as lakes and wooded streams appear on many advertisements chiefly to attract attention and to get the reader to read the rest of the advertisement, not necessarily to draw conclusions like those above. Yet, in many cases the advertiser hopes for the type of thinking here suggested.

Many of the conclusions of science are at first purely inductive in nature. A certain type of treatment for colds cures 80 cases out of 100. Cure is not certain; in fact, the probability is .8 that a patient will be cured. In cases involving human beings there are always so many possible varying factors that conclusions are often not certain. Yet, if one takes as a part of his premise this uncertainty, deductive reasoning is still possible. That is, if this treatment has a probability of cure equal to .8, then 8 out of 10 patients so treated will get well.

Much of the work in science and engineering can not take any such chances. A bridge is built. It is not sufficient if 8 out of 10 bridges so built have held up under the strain. Certain laws (postulates) have been discovered concerning stress and strain in a bridge. These have been discovered by inductive reasoning and established by experiment. Then to determine the size of various beams and supports these laws are applied and a margin of safety added. In fact, science is built up of laws experimentally determined, really postulates based on inductive thinking, then, accepting these as true, rigorous deductive reasoning is done.

In deductive reasoning one proves a general conclusion by supporting it with other generalizations previously accepted. The following are illustrative of deductive reasoning in non-geometric situations.

The thinking concerning killing mosquitoes by covering all standing water with a film of oil was deductive thinking. Note that one general statement was proved by showing its dependence on several others. The fact that mosquitoes lay their eggs in standing water is a postulate based on inductive thinking through observation. The fact that they hatch into larvae and breathe air may have been deductive reasoning based on the fact that a mosquoto is an insect and all insects live that way. The oil film preventing breathing was also probably based on observation of a few special cases and therefore is really a postulate based on induction also.

If forests are restored, floods will be largely controlled.

Premise: Forests are restored.

Conclusion: Floods will be largely controlled.

Proof: 1. The roots of trees penetrate deeply into the soil and tend to make the soil more porous.

2. The accumulation of leaves on the ground makes a deep mulch in a few years.

3. The loosened soil and the heavy mulch hold a great deal of water and therefore greatly retard the "run-off."

4. Rapid "run-off" causes floods by congestion of drainage outlets.

5. Therefore, if forests are restored, floods will be controlled.

If you buy a "Wazeek" electric, cabinet sewing machine, you will save money on your clothing bill.

Premise: You buy a "Wazeek" sewing machine.

Conclusion: You will save money.

Proof: (Such as typically given by many salesmen.)

1. Notice what a beautiful piece of furniture this machine is. It is a fitting decoration for any living room and serves charmingly as a flower stand.

2. Note how smoothly the machine runs, because of the fine workmanship and the rotary bobbin.

3. The special equipment is unparalled. It includes hemmers, ruffler, binder, tucker, and plaiter. These we shall not

only demonstrate but also teach you to use.

4. With a little experience in following patterns you can, with this splendid machine and its unusual equipment, make a fifty-dollar dress for from five to ten dollars, thus saving the profits of several dealers.

5. Just now we are giving a special offer on this latest "Wazeek" machine for the first five sold in the community. You have been highly recommended to us as a woman of influence and, therefore, we are anxious to place our first machine with you.

6. The cost is only ten dollars cash. The ten-dollar monthly installments will really cost you nothing because by using the machine you will save more than ten dollars a month.

7. What are Doctor Doe's initials? Would you prefer the mahogany or the walnut finish?

8. Now write your name on this line and you will have done a good day's business because you will have saved enough money to buy some of the extras you may have been longing for.

This is the type of argument that one frequently hears from a salesman. The several fallacies are glaring when the argument is written down in this form. Substitute vacuum cleaner, automobile, radio, brushes, a new set of golf clubs, or a correspondence course in engineering, and many of the same arguments will still be typical.

The following situations are suggestive of extensive use of deductive thinking in situations where certain conditions involve inevitable consequences. That is, they are "if-then" situations.

1. If the lower drafts on a stove are regulated, the fire is controlled.
2. If a boy having flat feet will take the proper exercises, he will be cured.
3. If the air is over saturated, rain will form.
4. If all means and routes of travel by which a disease-producing organism might pass from patient to victim are cut off, infectious diseases would never be contagious.
5. If the correct lens is worn before the eye, eyestrain is decreased.
6. If your specific gravity is greater than one, you will be unable to float in water.
7. If the valve is not properly seated, the compression will be poor.
8. If you can get the life history of an insect pest and destroy any link in the cycle, the pest will be controlled.
9. If at the age of 25 you save one hundred dollars every six months for thirty years and invest it at five per cent, then at the age of 55 you can buy an annuity of one hundred dollars per month for life.
10. If the head of your golf club comes in too slowly, you will get a slice, if too rapidly, a hook.

While deductive reasoning in non-geometric situations resembles in many ways the same reasoning in geometry, there are striking differences. The dependence of the conclusion upon the premise and upon other conclusions which may be thought of as postulates or previous theorems is of course exactly the same. The chief difference lies in the larger number and the complexity of both postulates and previous theorems, as for instance, those listed in the first illustration concerning mosquito control. Then, too, clear definitions of technical terms are needed, both in geometry and outside of geometry. They are often lacking in non-geometric reasoning. (Read Stuart Chase, "Tyranny of Words," November *Harper's Magazine*, also, Chase's article in the January *Harper's*.) Finally, converses and opposites in geometry are never accepted without proof. Unfortunately, this is not true in non-geometric thinking. Converses and opposites are too often assumed to be true.

Converses and Opposites in Science, Business, Industry, and Sports

Converses in non-geometric reasoning are very common and frequently assumed to be true without proof. The following illustrations are selected to emphasize this fact. Naturally, not all converses are false, although most of them probably are.

1. Faraday discovered that, "If a loop of wire is rotated through a magnetic field, then a current of electricity will flow." Conversely, Oersted reasoned (a) "If electricity is passed through wire wound around an iron core, it will produce a magnet," and (b) "If an electric current is passed through a wire in a magnetic field, motion in the wire is produced."

This statement and its two converses are really postulates. They are not proved deductively, but are accepted as a basis for work in electricity. Faraday's statement is the basis for the dynamo, upon the first converse the electro-magnet is based, and upon the second converse, the electric motor.

2. The advertiser who shows beautiful girls on a cigarette or liquor advertisement hopes the observer will not only generalize but also accept the opposite, the converse, and the opposite of the converse. Here they are.

a. *The generalization based on inductive thinking:* All beautiful girls smoke X cigarettes, or if you are beautiful you will smoke X cigarettes.
b. *The opposite:* If you are not beautiful, you will not smoke X cigarettes.
c. *The converse:* If you smoke X cigarettes, you will be beautiful.
d. *The opposite of the converse:* If you do not smoke X cigarettes, you will not be beautiful.

The effectiveness of these advertisements seems to suggest that many readers not only generalize but also often assume the opposite, the converse, or the opposite of the converse. Change the subject to face powder, soft drinks, tooth paste, sport goods, clothing, automobiles, or ocean cruises and you can find the same kind of so-called thinking. Why is that so many travel advertisements show a group of charming young people, and in addition, usually have more men than girls present? Is it not possible that the advertiser wishes the lonesome girl to assume that all cruises have more handsome men than beautiful girls, as well as to assume the converse and the opposite of the converse, that all beautiful girls go on ocean cruises and that no beautiful girl does not go on ocean cruises?

Converses need proof. They are usually not true, because there are other conditions that produce the same result. That is, the condition stated in the premise is sufficient to produce the result stated, but not necessary. A converse will be true only when the premise is necessary for the conclusion. For instance, right angles are equal, but equal angles are not always right angles. "Right angles" is not a necessary condition for equality, even though sufficient. However, if two arcs of the same circle are equal, their central angles are equal, and conversely. Here the hypothesis is not only sufficient but necessary.

If I have a tank full of water, from it I can fill a small pail. If I am rich, I can buy a postage stamp. If I am a man, I can lift 100 lb. If I am a gentleman, I will rise when a lady enters the room.

The converses of these statements are none of them true. The reason is quite evident. It is also evident that the conclusion could be changed so as to make the converse true: If I have a round tank full of water, from it I can fill a square tank of equal capacity, etc. A statement and its converse or opposite are both true only when the condition in the premise is *necessary* as well as *sufficient* to make the conclusion true.

Too often geometries never mention converses or opposites which are not true, and therefore probably mislead students into thinking that all converses and op-

posites are true. Yet, in geometry every converse or opposite is proved and therefore emphasis is placed (even though often inadequate) upon the necessity of proving such reverse statements.

Indirect Proof

Further study of converses would be most entertaining. They are often involved in indirect proof. In fact, one common form of indirect proof uses the converse statement in the process of exhausting the possibilities. Another form, probably the most useful one, consists in proving a statement true by proving its opposite false. For example, the accused man either committed the crime or he did not commit it. One of these statements is the opposite of the other. The car is either out of gas or not out of gas. The missing boy was either drowned or not drowned, kidnapped or not kidnapped, killed or not killed. If in any situation one of these two possibilities can be shown false, the other will be true without further reasoning. Hence the name, indirect proof.

One evening as Mr. Adams was sitting in his living room reading, his light went out and he was in darkness. What was the matter? Was the current off at the central station? He looked out of the window and found his neighbors with lights. Probably the bulb was burned out. He tried another light but with no success. Then he decided to try the fuse box. He replaced a fuse that looked black and his lights came on.

This illustration contains three examples of indirect proof, that is, proving the opposite impossible. First, the current was either off or not off at the central station. If off, then the neighbors would have no lights either. But since the neighbors had lights, the assumption that the current was off was impossible. Since there are only the two possibilities, one of which must be true, then the current is not off at the central station. Second, the bulb is either good or not good. If not good, than other bulbs could be lighted. But other bulbs do not light either, and it is unlikely that all the bulbs are burned out at the same time, so the assumption that the bulb is not good is false. Third, a fuse is either blown or not blown. If not blown, putting in a new fuse would not change matters. But putting in a new fuse does change matters, therefore, the old fuse was worthless.

The use of the indirect method is very common in non-geometric life situations. In fact, it is much more common than direct reasoning. Often in order to establish the cause of some accident or occurrence the indirect form is most effective. The man's death was either caused by poisoning or not. The car stops at the crossing or does not. The bank was robbed either by this man or not by this man. The reason the radio does not work is either because of defective tubes or some other reason.

Indirect reasoning sometimes takes a quite different form. It may not directly have anything to do with opposites. For instance, I lost my umbrella this morning. I may have left it on the street car, at the library, at the bank, at the drug store, at X's department store, or on the street car coming home. These are the places I visited. When I left the bank I remember it caught in the door so the first three are eliminated. When I left X's I had some packages and I do not remember carrying the umbrella. I think I left it either there or at the drug store. When I left the drug store, I met my good friend George and I recall that I pretended to strike him with it, so I must have left it at X's.

Note that in this form of indirect reasoning several possibilities are set up and then all but one eliminated. Deeper analysis will reveal that each possibility may be eliminated by the use of opposites. However, they are usually eliminated without thought of opposites.

Indirect reasoning is commonly used (1) in finding a book in a library, (2) in locating a boy in a large high school, (3) in finding an address in a classified di-

rectory, (4) in deciding on the meaning of a word in a certain setting, (5) in repairing a car or radio, (6) in planning a meal, (7) in finding John in the evening, (8) in deciding what bait to use in fishing, (9) or what club to use in golfing, (10) or what sort of a return to give your opponent in tennis, (11) or what sort of a ball to pitch to a certain batter in baseball, or (12) what play to use in football, or (13) what dress to wear to a party.

Summary

Careful thinking in any field of activity involves definitions of terms used, quite likely some postulates and previously proved conclusions, and frequently converses and opposites. Generalizing from a few cases (inductive thinking) is far too frequently done, and too seldom do we use rigorous deductive thinking. When we prove statements, or try to prove them, we often use direct proof and also indirect proof.

In non-geometric thinking the systems of postulates are sometimes complex and often not recognized as postulates. Conclusions are too often accepted without analysis of the bases upon which they rest. In geometry the postulates are clearly and simply stated. Proofs give the authority for every step. It is easy to find the statements upon which a proof depends.

Therefore, geometry illustrates clear thinking. In fact, it shows how thinking must be done if it is to be sound, dependable, rigorous. This paper has tried to show that geometry, while demonstrating many useful facts and ideas concerning size, shape, and position, also shows the way to do clear thinking.

Summer Meeting of the National Council!

The National Council of Teachers of Mathematics will have its fourth annual summer meeting with the N.E.A. on June 27, 28, and 29, in New York City. Monday afternoon there will be two sessions, one devoted to arithmetic in the elementary grades and the other a joint session with the Department of Secondary Education, devoted to the general topic, "The Forgotten Pupil." Tuesday and Wednesday afternoons will be devoted to current problems in connection with Junior High School and Senior High School.

A more detailed program will appear in the May issue of the "Mathematics Teacher."

Preliminary Report of the Joint Commission!

The Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics on "The Place of Mathematics in Secondary Education" is issuing a Preliminary Report in two parts. The first part has already been issued, and it is expected that the second part will appear about July 1st. The report is being issued in a preliminary form in order to secure the comments and suggestions of teachers, administrators, and educators. Persons desiring to receive copies can do so by sending ten cents (coins preferred) for each part to the Chairman, Professor K. P. Williams, Indiana University, Bloomington, Indiana.

The Importance of Certain Concepts and Laws of Logic for the Study and Teaching of Geometry*

By Nathan Lazar
Alexander Hamilton High School, Brooklyn, N. Y.

Chapter II

THE INVERSE

The Inverse (Opposite) of a Theorem

Although the definition of the term inverse (opposite) will be given more fully later in this chapter, it is desirable to present a few examples now so as to clarify the meaning of the term.

I. Theorem.
If two angles are equal, their supplements or complements are equal.
Inverse Theorem.
If two angles are unequal their supplements or complements are unequal.

II. Theorem.
If two angles are right angles, they are equal.
Inverse Theorem.
If two angles are not right angles, they are not equal.

III. Theorem.
If two sides of a triangle are equal, the angles opposite these sides are equal.
Inverse Theorem.
If two sides of a triangle are unequal, the angles opposite these sides are unequal.

IV. Theorem.
If a pair of alternate interior angles are equal, the lines are parallel.
Inverse Theorem.
If a pair of alternate interior angles are not equal, the lines are not parallel.

Occurrence of the Concept Opposite

Although the notion of converse appears in nearly all the textbooks on geometry examined, in only twelve is mention made of the concept of the opposite of a theorem. Moreover, in these twelve books this concept is not made an essential part of the logical or pedagogical scheme, but instead enters as an *arrière pensée* usually in connection with a problem on locus.

In the following list, which is by no means exhaustive, those textbooks are enumerated in which the writer found the term *opposite* or its equivalent mentioned. The order given is chronological.

Wilson, J. M. *Elementary Geometry*, p. 3. Macmillan and Company. 1881.
Halsted, G. B. *The Elements of Geometry*, p. 4. John Wiley and Sons. 1885.
Wentworth, G. A. *Plane Geometry*, p. 5. Ginn and Company. 1899.
Failor, I. N. *Plane and Solid Geometry*, p. 2. The Century Company. 1906.
Smith, E. R. *Plane Geometry Developed by the Syllabus Method*, p. 17. American Book Company. 1909.
Hart, C. A. and Feldman, D. D. *Plane Geometry*, p. 45. American Book Company. 1911.
Durell, F. *Plane and Solid Geometry*, pp. 24–25. Charles E. Merrill Company. 1912.
Auerbach, M. and Walsh, C. B. *Plane Geometry*, p. 155. J. B. Lippincott Company. 1920.
Bernard, D. M. *Plane Geometry*, p. 54. Johnson Publishing Company. 1927.
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* This is the second installment of Dr. Lazar's thesis. The first appeared in the March issue of *The Mathematics Teacher*. Preprints of this thesis (66 pages) *bound in cloth* may be obtained postpaid for $1 from *The Mathematics Teacher*, 525 W 120th St., New York City.

The Term Inverse Instead of Opposite

Although it is in general difficult and inadvisable to discard a technical term and substitute another for it, it is highly desirable to introduce the use of the term *inverse* instead of *opposite* for the following reasons:

1. The term opposite, which purports to described a *definite* logical operation, has *never* been used by logicians in that sense. The following quotation will show that it is used in a more general way:

Two propositions are technically said to be *opposed* to each other when they have the same subject and predicate respectively, but differ in quantity or quality or both.[1]

What the mathematician calls opposite theorems may be referred to by the logician by any one of four designations.

2. The word *opposite* is used in other contexts and therefore tends to mislead the student, because of a false feeling of familiarity with the meaning of the word.

3. The term *obverse* has been used in some textbooks[2] to refer to the transformation commonly called opposite. But that term is now generally used in standard books on logic in a different sense.

4. Since the word *inverse*, which is universally accepted among British and American logicians, resembles greatly the logical transformation called "the opposite" by mathematicians, there is no reason for the continued use of the latter term.

Definitions of the Opposite (Inverse)

Syllabus of the A.I.G.T. In the *Syllabus of Plane Geometry* prepared by the Association for the Improvement of Geometrical Teaching[3] the proposition

If A is not B, C is not D

is termed the opposite[4] of the typical theorem

If A is B, C is D.

In the textbook[5] based on that syllabus the term is mentioned a few times, but it does not enter into the plan of the book as integrally as the term converse. Thus, no theorem is termed the opposite of another theorem, as some theorems are called the converses of others. The definition is important however because it was adopted by many contemporary and later authors of textbooks.

Schultze. Schultze provides the following schematic illustration of the opposite of a theorem without analyzing the operation of getting it:[6]

If the theorem be represented by:

If A is B, then a is b;

its opposite would be:

If A is not B, then a is not b.

Christofferson. Christofferson, likewise, gives an example of the opposite of a theorem instead of a definition.[7]

The statement that all non-X is non-Y is the opposite of the statement that all X is Y. The theorem that all triangles with two equal sides have two equal angles, has for the opposite the statement that all triangles with "not two" (without two) equal sides have not two equal angles.

Auerbach and Walsh. In their *Plane Geometry* Auerbach and Walsh give the following definition:[8]

The opposite of a fact negates both the data and the conclusion.

[1] Keynes, J. N., *Studies and Exercises in Formal Logic*, p. 109. Fourth Edition. Macmillan and Co. 1906, 1928.

[2] Halsted, G. B., *The Elements of Geometry*, p. 5. John Wiley and Sons. 1885.

Wilson, J. M., *Elementary Geometry*, p. 3. Macmillan and Co. 1881.

[3] Association for the Improvement of Geometrical Teaching, *Syllabus of Plane Geometry*, p. 4. Macmillan and Company. 1875, 1889.

[4] The word actually used was *obverse*.

[5] Association for the Improvement of Geometrical Teaching, *The Elements of Plane Geometry*. Part I (corresponding to Euclid, Books I–II), Prepared by the Committee appointed by the Association, London: W. Swan Sonnerschein and Co., 1884. Part II (Corresponding to Euclid, Books III, IV, V, VI), appeared in 1886.

[6] Schultze, Arthur, *The Teaching of Mathematics in Secondary Schools*, p. 144. The Macmillan Co. 1912.

[7] Christofferson, H. C., *Geometry Professionalized for Teachers*, p. 131. Oxford, Ohio. 1933.

[8] Auerbach and Walsh, *op. cit.*, p. 155.

Hart and Feldman. The following is given as the definition of the opposite by Hart and Feldman:[9]

One theorem is the opposite of another when the hypothesis of the first is the contradiction of the hypothesis of the second, and the conclusion of the first is the contradiction of the conclusion of the second.

Comments on the Definition and Use of the Term

The last two definitions quoted are representative of those found in the other books mentioned. Strangely enough, some of the authors introduce the term without any explanation or definition whatsoever, and do not use it again after the topic of locus is completed.

Criticism of the Above Definitions

The definitions of the opposite (inverse) of a theorem enumerated above are unquestionably applicable to theorems that have but one condition in the hypothesis and one consequence in the conclusion, such as

1. Vertical angles are equal.
2. All right angles are equal.
3. If two sides of a triangle are equal, the angles opposite are equal.
4. If alternate interior angles are equal, the lines are parallel.

But what would be the opposite (inverse) of propositions that have two or more hypotheses and one conclusion? The following are examples:

5. If a line bisects the vertex angle of an isosceles triangle it bisects the base.
6. Every point in the perpendicular bisector of a line is equidistant from the ends of the line.
7. The straight line perpendicular to a radius at its outer extremity is a tangent to the circle.

A still more embarrassing question is, What, according to the definition, is the opposite of theorems that have two or more hypotheses and two or more conclusions?

8. If a radius is perpendicular to a chord it bisects the chord and its arc.
9. If the three sides of one triangle are equal respectively to the three sides of another triangle, the corresponding angles are equal.

The Spurious Simplicity of the Verbal Form of a Theorem

"But what is so difficult about it?" the reader is tempted to remonstrate. "Just change the verbs in the hypothesis and the conclusion to their negative forms and you get the opposite of the theorem." Thus the opposite of Theorem 5 is

If a line does not bisect the vertex angle of an isosceles triangle it does not bisect the base.

At first glance this inverse may appear clear, but a closer analysis will show it to be otherwise.

It is often believed that the verb of the "if" portion of a proposition states the facts to be considered as the hypothesis of that proposition. But that superficial view is due to the failure to realize that because of the structure of language it is possible to crowd many facts into one sentence.

Not until one actually has to prove the above theorem will one discover that the clause—"If a line bisects the vertex angle of an isosceles triangle"—really tells more than may be evident at first. To single out the verb "bisects," and to call it *the hypothesis* of the proposition, betray an unawareness of the pitfalls of language.

The following formulation of the same Theorem 5 will perhaps serve as a better proof of the inadequacy of the traditional definition of the opposite, and of the undependability of the verbal form of a proposition:

If the bisector of the vertex angle of an isosceles triangle is extended to the base, it will intersect it at its midpoint.

On the usual definition, "to deny both the hypothesis and the conclusion," the following opposite would result:

[9] Hart and Feldman, *op. cit.*, p. 45.

If the bisector of the vertex angle of an isosceles triangle is *not* extended to the base, it will *not* intersect it at its midpoint.

Formulated thus, the theorem is too absurdly obvious to deserve any serious consideration on the part of any student of geometry. Observe however the meaningful and definite opposites when the theorem is carefully analyzed:

Data	Conclusion
triangle ABC	1. AD bisects BC
1. $AB=AC$	
2. AD bisects angle A	

By contradicting the conclusion and *only one* of the hypotheses at one time, the following theorems result:

Theorem 5.1

Data	Conclusion
triangle ABC	AD does not bisect BC
1. $AB \neq AC$	
2. AD bisects angle A	

Theorem 5.2

Data	Conclusion
triangle ABC	AD does not bisect BC
1. $AB=AC$	
2. AD does not bisect angle A	

It will be noted that both opposites are meaningful and interesting theorems.[10]

Theorem 5.1 may be put into words as follows:

If a line bisects an angle of a triangle included between two unequal sides, it divides the third side into unequal segments.

This theorem is not only true but foreshadows the more definite statement of the relation of segments of a side of a triangle, formed by the bisector of the opposite angle, to the adjacent sides.

Theorem 5.2 affords interesting exploration into the possibilities of geometry and may serve as an excellent exercise in developing mathematical imagination.

Proposed Definition of an Inverse

The Inverse of a Proposition Having One Conclusion

The method by which the inverses of Theorem 5 were obtained leads the writer to propose the following definition of an inverse:

An inverse of a proposition having one conclusion may be formed by contradicting one of the hypotheses and the conclusion.

The Advantages of the Multi-Inverse Interpretation

1. The most important advantage of the multi-inverse approach is that it makes possible the extension of the concept of inverse to those theorems that have more than one condition in the hypothesis.

2. There is, moreover, an equality between the number of converses and the number of inverses that are derivable from a theorem with one conclusion.

Later it will be demonstrated that between the converses and inverses of a theorem there exists a logical relationship more important than mere numerical equality.

3. The inverses obtained in accordance with the definition advocated here do not depend on the verbal formulation of the proposition, but rather on the data and on the conclusion obtained through careful analysis.

4. Pedagogically it is of importance, for it enables the beginning student to reach out for himself and to explore geometrical realms unhampered by the continual prodding of the teacher. The discovery of new theorems, the persistent probing to discover their truth or falsity, the constant reminder that they have to be proved independently of the basic theorem, and, finally, the search for appropriate proofs, afford an unequalled opportunity for discipline in logic.

[10] It is of course unnecessary to remind the reader that the truth or falsity of the opposite of a theorem is *independent* of the truth or falsity of the theorem itself, and therefore requires independent proof.

Reason for Limitation of Inverse to Denial of Only One of the Hypotheses

The reader has undoubtedly wondered at the arbitrary limiting of the definition of an inverse to the denial of only one of the conditions in the hypothesis. The justification of that procedure can be made only after an analysis of the function of a definition.[11] Without going too deeply into that problem it is evident that only those transformations of geometric propositions should be considered that yield, occasionally at least, true propositions. The geometer is as interested in discovering new theorems as the scientist is in discovering new facts. The mathematician uses logical transformations in the same way that the scientist uses new hypotheses. Thus the reason for the important rôle played by converses of a theorem in the study of mathematics lies in the fact that many of the converses investigated turn out to be true.

Many other possible transformations studied in great detail in logic are not dealt with in mathematics because of the small yield of significant or true theorems. It is, for example, possible to obtain from every theorem a second theorem, in which one of the hypotheses is the contradictory of the first but the conclusion remains the same. Thus, the theorems

All vertical angles are equal.
The base angles of an isosceles triangle are equal.

would yield the following transformations:

All angles that are not vertical are equal.
If a triangle is not isosceles, the base angles are equal.

These theorems are false, and so will be the majority of similar transformations of true theorems. It is for that reason that no mathematician will trouble himself with such possible transformations.

[11] The reader interested in this problem will find a discussion in Cohen, M. R. and Nagel, E., *An Introduction to Logic and Scientific Method*, pp. 224–233. Harcourt, Brace and Co. 1934.

In the problem of choosing a definition for the inverse of a proposition, the guiding principle should be that the one to be selected should yield a high percentage of true transformations. It is the contention of the author that the definition advocated here possesses that property to a greater degree than the other possible definitions. This claim can best be illustrated, though not demonstrated, by actual examples.

An Alternative Definition of Inverse

Another possible definition of an inverse would be the following:

An inverse of a theorem having one conclusion may be obtained by denying the conclusion and any number of the conditions in the hypothesis.

The application of this definition to the following theorem will exhibit its potentialities:

Every point in the perpendicular bisector of a line is equidistant from the ends of the line.

Data	Conclusion
1. $CD \perp AB$	$PA = PB$
2. CD bisects AB	
3. P is on CD	

By denying only one of the data at a time and the conclusion, the following three inverses result:

Inverse 1

Data	Conclusion
1. CD is not $\perp AB$	$PA \neq PB$
2. CD bisects AB	
3. P is on CD	

Inverse 2

Data	Conclusion
1. $CD \perp AB$	$PA \neq PB$
2. CD does not bisect AB	
3. P is on CD	

Inverse 3

Data	Conclusion
1. $CD \perp AB$	$PA \neq PB$
2. CD bisects AB	
3. P is not on CD	

By denying at the same time two of the data and the conclusion, the following 3 inverses result:

Inverse 4

Data	Conclusion
1. CD is not $\perp AB$	$PA \neq PB$
2. CD does not bisect AB	
3. P is on CD	

Inverse 5

Data	Conclusion
1. CD is not $\perp AB$	$PA \neq PB$
2. CD bisects AB	
3. P is not on CD	

Inverse 6

Data	Conclusion
1. $CD \perp AB$	$PA \neq PB$
2. CD does not bisect AB	
3. P is not on CD	

The following inverse is obtained by denying all the data, and the conclusion:

Inverse 7

Data	Conclusion
1. CD is not $\perp AB$	$PA \neq PB$
2. CD does not bisect AB	
3. P is not on CD	

A few minutes with pencil and paper will convince the reader of the truth of the following statements:

1. Inverses 2 and 3 are true. They were obtained in accordance with the definition advocated above—by denying the conclusion and *only one* of the data.

2. The inverses obtained by denying two or more hypotheses are all false. No claim is made, however, that similar results will be obtained in transforming other theorems, since it is a well-known fact that there are many geometric propositions that have false inverses although only one of the data is denied. A sufficient number of theorems and their inverses have already been examined to warrant the acceptance of the definition advocated in this study as the one most likely to yield a greater number of true inverses.

Another possible definition of the inverse of a theorem. It is also possible to define the inverse of a theorem in the following manner:

An inverse of a theorem may be obtained by denying at least one of the hypotheses and at least one of the conclusions.

Such a definition would yield a great number of propositions which are interesting, meaningful, and true. It is, however, evident that the form of such propositions is not the one that is usually encountered in mathematics in general and in geometry in particular. It is not then the definition that will be adopted in this study.

The Inverse of a Theorem Having More Than One Conclusion

If a theorem has more than one datum in its hypothesis and more than one statement in its conclusion, it might seem that, by analogy with the converse, a denial of one of the hypotheses would necessitate a denial of only *one* of the conclusions. Unfortunately, such is rarely the case. In many of the true inverses examined, a denial of one of the data is followed by the denial of the majority or of *all* the conclusions. Thus, in Theorem C on page 109, if either datum 1 or datum 2 is denied, *both conclusions* must be denied in order to obtain a true proposition. Again, in Theorem D on page 110, if datum 1 is denied, i.e., if AB is not parallel to DC but datum 2 is retained unchanged, the following will result: (a) the first conclusion, $AB = DC$ will be true only in the special case when $ABCD$ is an isosceles trapezoid; (b) the conclusions 2, 3, and 4 will all have to be denied to obtain a true proposition.

The two illustrations mentioned above are typical of a great number of theorems and their inverses. Because of this lack of uniform behavior of the inverses of theorems having more than one conclusion, the recommendation is here made that in the elementary course in geometry the phrase "inverse of a proposition" should be applied only to those theorems that have not more than one conclusion.

This limitation of the concept is really of little consequence to the mathematician. Every theorem that contains many conclusions can always be subdivided into as many theorems as there are conclusions, all having the same set of hypotheses. Each one of the resulting theorems will then yield as many inverses as there are data in its hypothesis. The sum total of

the inverses of these theorems may be considered as belonging to the original theorem with the multiple conclusions.

Pedagogical Importance of the Inverse

The full logical importance of the inverse cannot be expounded at this point. In the following chapters, after the introduction of the Law of Contraposition, it will be proved that a converse and its corresponding inverse are two different modes of stating the same fact. It is, however, important to keep in mind that although the general tendency among teachers and textbooks is to point out the fallacy of conversion and warn against it, it is not as common an error as the corresponding fallacy of inversion.

Consider for example an advertising slogan like (I) "An Olian watch is a good timepiece." The advertiser is not only interested in stating a fact about this product, but also hopes that the reader or the listener will make the following inference: (II) "If the watch is not an Olian, it is not a good timepiece."

It should be noticed that the second inference is the inverse of the first. Since the inferring of an inverse is an invalid process of reasoning, the second would not necessarily be true even if the first were. Thus the stressing of the invalidity of the *inverse* of a proposition in geometry, followed by illustrations taken from other fields, may help to inculcate a valuable logical habit that may persist in the pupil's mind long after the theorems about angles, lines, and areas have been forgotten.

Summary

In this chapter the discussion of logical concepts was continued by an examination of the definition of the opposite of a theorem. The comparative rarity with which this definition appears in textbooks was pointed out, with the consequent loss of many opportunities in the training of logical insight. The use of the term *inverse* was advocated in place of the more common term *opposite*. Attention was called to the defects in the traditional use of the term *opposite*, and a substitute definition was proposed. The advantages of the latter definition have been discussed in detail.

Chapter III

THE CONTRAPOSITIVE

The Meaning of the Term

It is an inexplicable phenomenon in the history of the teaching of mathematics and of logic that although students of those subjects are usually conversant with the converse and the inverse of a proposition which are *not* valid inferences, comparatively few students are familiar with another transformation—the contrapositive—which *always* yields valid inferences. But before entering into a detailed study of the history and definition of the contrapositive of a theorem it is desirable to acquaint the reader with the meaning of the term by means of a few illustrations.[1]

I. Theorem.
If two angles are equal, their supplements are equal.
Contrapositive Theorem.
If the supplements of two angles are not equal, the angles are unequal.

II. Theorem.
If two angles are right angles, they are equal.
Contrapositive Theorem.
If two angles are not equal they are not right angles.

III. Theorem.
If two sides of a triangle are equal, the angles opposite these sides are equal.
Contrapositive Theorem.
If two angles of a triangle are un-

[1] The theorems used here were also mentioned on page 156, in connection with the subject of inverses.

equal, the sides opposite these angles are unequal.

IV. Theorem.
If a pair of alternate interior angles are equal, the lines are parallel.
Contrapositive Theorem.
If two lines are not parallel, the alternate interior angles are not equal.

Other Names for the Contrapositive

Although the transformation illustrated above is at present generally called *the contrapositive* by British and American logicians, the following designations for the same process have also been used: contraversion,[2] contranomial,[3] converse of opposite,[4] opposite of converse,[5] and converse of a contrary (contrarie conversum).[6] In this study the terms "contraposition," "contrapositive statement," or simply "contrapositive" will be used interchangeably, depending upon the context.

Historical Survey

A brief historical survey of the concept of contrapositive and of the Law of Contraposition in relation to mathematics is set forth in the following paragraphs.[7]

Euclid. Since the concepts of converse and inverse do not appear in the *Elements of Euclid* it is not surprising that the notion of contrapositive is also missing from that classic in geometry.

Proclus. Proclus' commentary on Euclid, which pointed out the possibility that some theorems may yield two converses, does not, however, contain any reference to the process of contraposition, although it has been known since the time of Aristotle.[8]

[2] Keynes, J. N., *Studies and Exercises in Formal Logic*, p. 134. Fourth Edition. Macmillan and Company. 1906, 1928.

[3] Halsted, G. B., *The Elements of Geometry*, p. 4. John Wiley and Sons. 1885.

[4] Schultze, Arthur., *The Teaching of Mathematics in Secondary Schools*, p. 219. The Macmillan Company. 1912.

[5] Christofferson, H. C., *Geometry Professionalized for Teachers*, p. 131. Oxford, Ohio. 1933.

[6] Hauber, F. C. *Scholae Logico-Mathematicae*, p. 262. Reutlingae. 1829.

[7] See note 2 on page 101.

[8] Aristotle, *Topica*, Book II, chap. 8.

Hauber. The first author to recognize the possibilities of the contrapositive for geometry was Karl Friedrich Hauber (1775–1851) known by his Latin name Friderico Carolo Hauber. In his little known book *Scholae Logico-Mathematicae,*[9] published in 1829, he proves with the help of appropriate quotations from Aristotle, that if a theorem is true, its contrapositive is also true.

Granted if *A*, then *B*, it follows that if non-*B*, then non-*A*. (. . . posito, si *A* est, est et *B;* sequeter: si non est *B*, non etiam est *A*.)[10]

Unfortunately he did not fully realize the wide possibilities of that fruitful logical transformation. He devoted himself instead to the investigation of those theorems that can be proved by means of Euclid's fourth theorem, and to an enumeration of their converses, inverses, and contrapositives.

Drobisch. In his textbook on logic, Drobisch[11] quotes Hauber's Law of Converses and indicates many theorems in geometry that may be proved by its application.

Matzka. In 1845 an article appeared by Wilhelm Matzka[12] in which he pointed out, among other things, that many theorems in geometry which are proved separately are really contrapositives of each other, and are therefore logically equivalent to the other statements. If the proof of one is obtained by the usual geometric methods, the truth of the others need not be established independently, but may be established by *quoting* the Law of Contraposition.[13]

Delboeuf. In a book published in 1880, Delboeuf[14] quoted Hauber and Drobisch,

[9] Hauber, *op. cit.*, pp. 176, 207, 262–263.

[10] *Ibid.*, p. 176.

[11] Drobisch, Mortiz Wilhelm, *Neue Darstellung der Logik*, pp. 162–163. First Edition. 1836. Pp. 234–239. Fifth Edition. 1887.

[12] Matzka, Wilhelm, "Betrachtungen einiger Gegenstände der Logik, mit bensonderer Rücksicht auf ihre Anwendung in der Mathematik," *Archiv der Mathematik und Physik*, (J. A. Grunert, Editor). Series I, Vol. 6., pp. 353–369. 1845.

[13] *Ibid.*, p. 356.

[14] Delboeuf, J., *Prolégomènes Philosophiques*

and also alluded to the equivalence of a theorem and its contrapositive. He also gave a few illustrations from geometry and astronomy.

De Morgan. The first author to write in English about the use of the Law of Contraposition for the elimination of proofs of many geometric theorems, was, appropriately enough, the British logician and mathematician, Augustus De Morgan (1806–1871). In an article on Euclid, he wrote as follows:

> . . . [In Euclid's Elements] there is no distinction between propositions which require demonstration, and those which a logician would see to be nothing but different modes of stating a preceding proposition. When Euclid has proved that everything which is not A is not B, he does not hold himself entitled to infer that every B is A, though the two propositions are identically the same. Thus having shewn that every point of a circle which is not the centre is not one from which three equal straight lines can be drawn, he cannot infer that any point from which three equal straight lines are drawn is the centre, but has need of a new demonstration.[15]

In another place, he wrote:

> Euclid may have been ignorant of the identity of "Every X is Y" and "Every not-Y is not-X," for anything that appears in his writings; he makes the one follow from the other by new proof each time.[16]

In *Short Supplementary Remarks on the First Six Books of Euclid's Elements*[17] the following notes appear:

(1) I.27 is a logical equivalent of what is already in I.16.[18]

In Euclid's *Elements* I.27 is the following theorem:

If a straight line falling on two other straight lines makes the alternate angles equal to one another, the two straight lines shall be parallel to one another.

I.16 is the following theorem:

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

(2) I.30 is a logical equivalent of [a] more simple postulate.[18]

I.30 is the theorem:

Straight lines which are parallel to the same straight line are parallel to each other.

The more "simple postulate" referred to by De Morgan is the following: If two right lines coincide in two points, they coincide when produced.

A careful analysis will show, however, that I.30 can be recognized more easily than the above as the logical equivalent of the following postulate:

Two intersecting lines cannot both be parallel to the same straight line.

(3) III.9 [is] a logical equivalent of part of III.7; when it is proved that every *non*-central point is *not* a point from which three equal straight lines can be drawn, then Theorem III.9 is also proved.[19]

III.9 is the following theorem:

If a point be taken within a circle from which there fall more than two equal straight lines to the circumference, that point is the center of the circle.

The part of III.7 referred to above is the following:

From the same point [which is not the center of the circle] there can be drawn to the circumference two straight lines, and only two, which are equal to one another.[20]

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de la Géométrie et Solution des Postulats, pp. 88–91. Liége. 1860. The contrapositive is referred to as *inverse de la réciproque* or *réciproque de l'inverse.*

[15] De Morgan, Augustus, article "Eucleides" in *Dictionary of Greek and Roman Biography and Mythology*. Edited by William Smith. Vol. 2, p. 65. Boston: Little Brown and Company. 1859.

[16] De Morgan, Augustus, *Syllabus of a Proposed System of Logic*, p. 32. London: Walton and Maberly. 1860.

[17] *The Companion to the Almanac*, or *The Yearbook of General Information for 1849* (often bound in one volume with the *British Almanac of the Society for the Diffusion of Useful Knowledge*, for 1849). Pp. 5–20. London: Charles Knight.

[18] *Ibid.*, p. 8.

[19] *Ibid.*, p. 10.

[20] The theorems are quoted (with only a slight omission in III. 7) from the *Elements of Euclid*, Edited by Isaac Todhunter, with an

The logical equivalence to which De Morgan refers is, of course, the equivalence of a theorem and its contrapositive, or in De Morgan's words, "The identity of Every X is Y, and Every not-Y is not-X."

The Influence of De Morgan

The prestige and eminence of De Morgan as logician, mathematician, and educator undoubtedly helped to popularize the idea of the contrapositive and to accelerate its inclusion in textbooks during the second half of the nineteenth century.

British Textbooks Containing the Notion of Contraposition

Wright. Although it was impossible to obtain for examination a copy of Wright's *Elements of Plane Geometry*,[21] it is evident from the criticism that appeared immediately after the publication of the book[22] that the idea of contraposition was at least mentioned in it. It is of course impossible to determine to what degree the author utilized the logical concept in the proofs of the theorems.

Hirst. As a disciple and admirer of De Morgan, T. A. Hirst (1830–1892) was perhaps more directly responsible than his illustrious teacher for the infiltration of logical ideas into the teaching of geometry. He contributed the preface to Wright's *Elements of Plane Geometry*. He was the first president of the Association for the Improvement of Geometrical Teaching[23] and in his presidential addresses lost no opportunity to point out the significant rôle that logic may be made to play in the teaching of geometry. He said:

> The interdependence of geometrical propositions above alluded to, as one of the subjects to which teachers should habitually direct the attention of their pupils, is mainly logical in character, but nevertheless most essential to geometrical culture. Every one will admit the primary importance of habituating the student to extract its full logical significance from every proposition he establishes, to recognize each proposition readily under different, although logically equivalent forms of enunciation, and thus to discriminate accurately between the cases where mere *logical* deduction from antecedent propositions is requisite, from those which require the introduction of further *geometrical* considerations. Obvious as this may be it is rarely sufficiently attended to by teachers, and even in approved textbooks, ancient as well as modern, we not unfrequently find remarkable instances of the absence of the discrimination to which I refer. The ninth proposition of the third book of Euclid is now a well-known case of the kind. Geometrical apparatus is there employed to demonstrate, indirectly, what had virtually been already proved in the seventh proposition. Having proved that *from a point which is not the centre three equal straight lines cannot be drawn to the circumference of a circle* (Prop. 7), it was wholly unnecessary to prove that *the point from which three equal straight lines can be drawn to the circumference must be the centre of the circle* (Prop. 9).
>
> The two theorems are, in fact, contrapositive forms, one of the other; the truth of each is implied, when that of the other is asserted, and to demonstrate both geometrically is more than superfluous; it is a mistake, since the true relation between the two is thereby masked. There can be no better proof of this than the fact that the above defect in exposition remained undetected for centuries. Another, though less striking, example of the same kind is presented by the 16th and 27th propositions of the first book. Few intelligent boys fail on first reading the 27th to note the oddity of giving to two parallel lines a dagger-like shape in order to prove indirectly that "If a straight line falling on two other straight lines make the alternate angles equal to each other; these two straight lines shall be parallel." It is certain, however, that few of them ever discover that the proposition has virtually been proved before, that it is in fact the contra-positive form of the 16th, since the latter is obviously susceptible of being thus enunciated: "If two straight lines meet one another, a straight line falling on them will not make the alternate angles equal."

Introduction by Sir Thomas L. Heath. Everyman's Library, No. 891. E. P. Dutton and Co. 1933.

[21] Wright, R. P. *The Elements of Plane Geometry.* Preface by T. A. Hirst. Longmans. 1868.

[22] *The Atheneum*, January 9, 1869, pp. 45–46.

[23] Often referred to as A.I.G.T.

The late Professor De Morgan, to whose keen penetration we owe the detection, not merely of the above defects in Euclid, but of many others, strongly and justly insisted upon the necessity of a more logical study of the elements of geometry.[24]

Hirst was, moreover, a member of the committee[25] to draw up a *Syllabus of Plane Geometry* for the Association, and was responsible for the Logical Introduction in which he expounded the logical concepts and laws with which he thought students of geometry should be familiar.[26]

The following selection from the Logical Introduction in the *Syllabus* is undoubtedly responsible for the interest that was shown, for one generation at least, in contraposition as a method of proof.

The enunciation of a Theorem consists of two parts, the *hypothesis*, or that which is assumed, and the *conclusion*, or that which is asserted to follow therefrom. Thus in the typical Theorem

If A is B, then C is D (i),

the hypothesis is that A is B, and the conclusion, that C is D.
From this Theorem it necessarily follows that:

If C is not D, then A is not B (ii).

Two such Theorems as (i) and (ii) are said to be *contra-positive*, each of the other. Two Theorems are said to be converse, each of the other, when the hypothesis of each is the conclusion of the other. Thus,

If C is D, then A is B (iii)

is the converse of the typical Theorem (i). The contrapositive of the last Theorem, viz.:

If A is not B, then C is not D, (iv)

is termed the *obverse* of the typical Theorem (i). Sometimes the hypothesis of a Theorem is complex, i.e. consists of several distinct hypotheses; in this case every Theorem formed by interchanging the conclusion and one of the hypotheses is a converse of the original Theorem. The truth of a converse is not a logical consequence of the truth of the original Theorem, but requires independent investigation.

Hence the four associated Theorems (i), (ii), (iii), (iv) resolve themselves into two Theorems that are independent of one another, and two others that are always and necessarily true if the former are true; consequently it will never be necessary to demonstrate geometrically more than two of the four Theorems, care being taken that the two selected are not contrapositive each of the other.[27]

Wilson. In 1881, J. M. Wilson[28] published a textbook on geometry in which he followed the *Syllabus of Geometry* prepared by the A.I.G.T. The book was an important contribution to the literature of the revolt against Euclid that was stirring in England in those decades, but it did not do much to further the cause of a purely logical approach to some of Euclid's theorems.

The Textbook of the A.I.G.T. In response to a demand from its members, the A.I.G.T. published a textbook in 1881–86 embodying the ideals and recommendations of the *Syllabus*. Although the technique of proof by the Law of Contraposition was occasionally used, it did not enter into the actual structure of the book either logically or pedagogically. A magnificent opportunity was thus lost for the introduction of logical patterns into geometry.

Henrici. In a small book on geometry Henrici[29] made a *Digression on Logic*, in which he pointed out various logical relations that may exist between propositions. After giving an example of a theorem and its contrapositive, he made the following interesting observation:

Though both forms express the same fact it is nevertheless often of importance to consider both. The contra-positive form often puts the truth expressed in a different light, so that the full meaning of the

[24] A.I.G.T., *Second Annual Report*, pp. 14–15. 1872.

[25] A.I.G.T., *Third Report*, p. 12. 1873.

[26] A.I.G.T., *Eighteenth Report*, p. 62. 1892.

[27] A.I.G.T., *Syllabus of Plane Geometry*, (corresponding to Euclid, Books I–VI). First published in 1875. Above quotation from New Edition, p. 4. Macmillan and Co. 1889.

[28] Wilson, J. M., *Elementary Geometry*, New Edition. Macmillan and Co. 1881.

[29] Henrici, O., *Elementary Geometry*, pp. 29–35. Second Edition. Longmans, Green and Co. 1888.

statement made may be more easily comprehended.[30]

He did not, however, utilize the equivalence between a theorem and its contrapositive to replace some traditional geometric proofs by logical ones.

American Textbooks Containing the Notion of Contraposition

The books enumerated chronologically below mentioned the concept of contraposition, but did not realize its possibilities:

Dupuis, N. F. *Elementary Synthetic Geometry*, pp. 2–3. Macmillan Co. 1889.

Smith, W. B. *Introductory Modern Geometry*, p. 40. Macmillan and Company. 1893.

Gillett, J. A. *Euclidean Geometry*, pp. 22–23. Henry Holt and Co. 1896.

Keigwin, H. W. *The Elements of Geometry*, p. 46. Henry Holt and Co. (Second Edition Revised.) 1898.

McMahon, James. *Elementary Geometry*, p. 50. American Book Co. 1903.

Smith, E. R. *Plane Geometry Developed by the Syllabus Method*, p. 17. American Book Company. 1909.

Strader, W. W. and Rhoads, L. D. *Plane Geometry*, p. 303. John C. Winston Co. 1927.

Halsted. The only textbook, within the knowledge of the writer, that realized in some degree the possibilities of the contrapositive transformation is the one written by George Bruce Halsted, under the title, *The Elements of Geometry.*[31]

Chapter I of Book I was headed "On Logic" and was devoted to a consideration of the following topics:

I. Definitions—Statements
II. Definitions—Classes
III. The Universe of Discourse
IV. Contranomial, Converse, Inverse, Obverse.[32]
V. On Theorems
VI. On Proving Inverses

On page 5, the following appears:

If the original statement is X is Y, its [contrapositive] is non-Y is non-X, its [converse] is Y is X, its [inverse] is non-X is non-Y. The first two are equivalent, and the last two are equivalent.

Thus, of four such associated theorems, it will never be necessary to demonstrate more than two, care being taken that the two selected are not [contrapositives].

From the truth of either of two [converses] that of the other cannot be inferred.[33]

Unlike the authors of the textbooks mentioned, Halsted used the equivalence of contrapositives to prove many theorems, and omitted the traditional geometric proofs.

The following proof of a well-known theorem is typical of the technique employed and is therefore reproduced here in full:

If two lines cut by a transversal make alternate interior angles equal, the lines are parallel.

This is the [contra-positive of Theorem X] part of which may be stated thus: If two lines which meet are cut by a transversal, their alternate interior angles are unequal.[34]

Theorem X states: An exterior angle of a triangle is greater than either remote interior angle.

On page 50, the theorem

If a transversal cuts two parallels, the alternate angles are equal,

is proved geometrically. But the theorem

If the alternate angles are unequal, the lines meet,

is not proved at all, but is merely designated as the contrapositive of Theorem XX and therefore true.

Similar uses of the Law of the Contraposition may be found on pages 120, 125, 145, 217.

Criticism of Halsted. Although Halsted

[30] *Ibid.*, pp. 30–31.

[31] Halsted, George Bruce, *The Elements of Geometry.* John Wiley and Sons. 1885.

[32] Halsted used the terms in senses different from those generally accepted now. Thus by contranomial he meant contrapositive. The term *inverse* he used where we would use *converse*, and *obverse* where we would use *opposite* or *inverse*. The term *converse* he restricted to the *valid* logical converse. Thus "Some y is x is called the logical converse of x is y."

[33] In order to prevent confusion of terminology, the present writer substituted the terms in brackets for those used by Halsted.

[34] Halsted, *op. cit.*, p. 48.

utilized the power of the Law of Contraposition to a greater degree than any of his forerunners or contemporaries, he was, nevertheless, prevented from obtaining a fuller insight into its potentialities by a narrow interpretation of the concept of a contrapositive and of the Law of Contraposition, which was common in his day because of the undeveloped state of Symbolic Logic. This criticism of Halsted is of basic importance and will be discussed at length in the next section.

Patterns of the Contrapositive

Strangely enough very few of the writers on mathematics stated the definition of a contrapositive. They merely gave the symbolized form of a statement and its contrapositive and then illustrated it with appropriate geometric theorems.

The symbolized patterns of a theorem and its contrapositive were usually cast in one of the two following forms:

1. Theorem: Every X is Y
   Contrapositive: Every non-Y is non-X:
2. Theorem: If A is B, then C is D
   Contrapositive: If C is not D, then A is not B.[35]

Thus the theorem,

Vertical angles are equal

would be interpreted in accordance with pattern 1, as follows:

Theorem: Every pair of vertical angles is a pair of equal angles.
Contrapositive: Every pair of non-equal angles, is a pair of non-vertical angles.

Following the pattern of 2, the above theorem and its contrapositive would be formulated as follows:

Theorem: If two angles are vertical, they are equal.
Contrapositive: If two angles are not equal, they are not vertical.

Defect of the Above Forms

The defect of the first form is its artificiality. It was taken over from the classic logic—the subject-predicate logic—which believed that every statement can be interpreted as ascribing an attribute to a subject.

It is generally accepted by modern logicians that

> . . . the subject-predicate logic is not full grown. Choosing the least complex of all possible forms of proposition—one that ascribes an attribute to a subject—it attempts to pour all other propositional forms into this simple mould and thence to construct all types of demonstration. A fragment of logic is mistaken for the whole.[36]

To be sure, any geometrical proposition can be forced, with a little ingenuity, into the Procrustean bed of

All X is Y,

but the unnaturalness of expression and the resulting obfuscation of meaning have unquestionably contributed to the general disrepute into which logic has fallen.

The second form,

If A is B, then C is D,

is highly suitable for the expression of mathematical laws in general and of geometric theorems in particular. The hypothetical character of the mathematical generalizations is thus brought into relief. Moreover the hypothesis is clearly distinguished from the conclusion—a characteristic very desirable from a pedagogical point of view.

Both forms have had, however, the pernicious effect of making generations of mathematicians overlook the fact that most geometric theorems have *more than one* condition in the hypothesis, a form to which the above patterns bear no resemblance. The facile way of stating the contrapositive in either of the two forms,

All non-Y is non-X,

or

If C is not D, then A is not B,

has also contributed to the same oversight.

[35] See p. 166.

[36] Eaton, Ralph M., *General Logic*, p. 66. Charles Scribner's Sons. 1931. See also Stebbing, L. S., *A Modern Introduction to Logic*, pp. 165–166. Thomas Y. Crowell Co. 1931.

When a theorem has but one hypothesis and one conclusion the above patterns, especially the second, yield the corresponding contrapositive very readily. Thus the theorem

If two sides of a triangle are equal, the angles opposite are equal.

gives rise without any difficulty to the contrapositive

If two angles of a triangle are not equal, the opposite sides are not equal.

Similarly, the theorem

If two lines are parallel, the alternate interior angles are equal

yields the equivalent contrapositive theorem

If alternate interior angles are not equal, the lines are not parallel.

The above patterns do not however enable one to formulate the contrapositive of the following familiar theorem:

If two sides of a triangle are equal, the line that bisects the included angle also bisects the third side.

Although the above theorem seems to divide itself naturally into two parts—the hypothesis and the conclusion—closer inspection will reveal the error. For, the hypothesis contains not only the information given in the clause "If two sides of a triangle are equal," but also the information in the phrase "the bisector of the included angle." The conclusion, moreover, does not begin as the verbal pattern indicates, with the words "The line that bisects the included angle . . . " but consists entirely of the last part, "bisects the third side." A detailed analysis will make this point clearer.

Data	Conclusion
triangle ABC[37]	AD bisects BC
1. $AB=AC$	
2. AD bisects angle A	

The above analysis gives a truer picture of the elements of the theorem and its meaning than either of the two patterns. It certainly does not fit into the mold

All X is Y

without doing violence to the clarity of the theorem.

The pattern of the contrapositive

All non-Y is non-X

is likewise of little value in formulating the contrapositive of the above theorem.[38]

The second pattern

If A is B, then C is D

is undoubtedly superior to the first as a general method of analyzing geometric propositions, but it, too, falls short of completeness, for it does not enable us to analyze theorems such as the above that have more than one condition in the hypothesis.

The second contrapositive pattern

If C is not D, then A is not B

is also of little help, since the hypothesis consists of two conditions, and this form does not specify the exact manner of transforming such a theorem into its equivalent contrapositive. If we obey the impulse to follow the above pattern implicitly, denying the hypothesis and the conclusion and interchanging them, we obtain the following contrapositive:

If a line does *not* bisect one of the sides of a triangle, it does *not* bisect the angle included between the two other sides, and the two other sides are unequal.

In terms of our customary analysis, this contrapositive may be stated thus:

Data	Conclusions
triangle ABC	1. $AB \neq AC$
1. AD does not bisect BC	2. AD does not bisect angle A

But as the following analysis will show, the above contrapositive is false. The two

[37] Note that in our analysis of a theorem, the specification that ABC is a triangle is mentioned but is not enumerated as one of the data.

[38] It is suggested that the reader attempt to formulate both the theorem and its contrapositive, following the pattern given immediately above.

following theorems contain the same data as the above contrapositive, as well as other data which are not inconsistent with them. But in each case one of the two conclusions is the contradictory of one of the above. In diagram 3a,

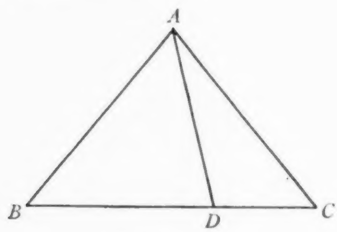

FIG. 3a.

Data
  triangle ABC
1. $AB=AC$
2. AD does not bisect BC

the conclusion, AD does not bisect angle A is true, but the first conclusion, $AB\neq AC$ is false, by hypothesis.

In the diagram below,

FIG. 3b.

Data
  triangle ABC
1. $AB\neq AC$
2. AD bisects angle A, and
3. AD does not bisect BC.

assumptions 1 and 2 are consistent with assumption 3. Since the bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides,

therefore $$\frac{BD}{DC}=\frac{AB}{AC}$$

but $AB\neq AC$,
therefore $BD\neq DC$.

The conclusion, $AB\neq AC$, is true by hypothesis but the second conclusion, that AD does not bisect angle A, is false by hypothesis.

The Dilemma

The false contrapositive exhibited above must lead us to one of two conclusions:

1. The Law of Contraposition is applicable only to theorems having one hypothesis and one conclusion.

2. The process of contraposition may be so analyzed and defined as to be applicable to any type of theorem regardless of the number of hypotheses or the number of conclusions it contains.

Fortunately the latter of the statements is true. But before it may be shown to be true, it is necessary to develop the following:

1. The statement and proof of the Law of Contraposition.

2. An analysis of the technical meaning of the phrase "contradictory of a statement."

Definition of the Contrapositive of a Proposition

The definition offered here is the one implied in the examples and illustrations quoted in the early part of this chapter:

The contrapositive of a theorem is another theorem obtained from it by contradicting both the hypothesis and the conclusion and by interchanging their respective places.

No limitations are set, however, on the number of data in the hypothesis or on the number of statements in the conclusion.

Statement and Proof of the Law of Contraposition

The law states that a proposition and the contrapositive derived from it are equivalent—if one is true, the other is true, and if one is false, the other is false.

If a proposition is symbolized.

$$h\rightarrow c$$

its contrapositive may be symbolized,

$$\bar{c}\rightarrow\bar{h},$$

where $\bar{c}$ means the *denial* or *falsity of the conclusion* and $\bar{h}$ means the *denial* or *fal-*

sity of the hypothesis, and $\rightarrow$ means *implies*.[39]

The method of deriving the proof of the law can be indicated best by a preliminary analysis of a specific example. Consider the following theorem:

Theorem I. If a triangle is isosceles, its base angles are equal.

Asserting a proposition means that the truth of the hypothesis involves the truth of the conclusion, i.e., that it is false to assert that the hypothesis is true and the conclusion is false. In terms of our illustration, Theorem I has the same meaning as the following:

Theorem II. The statement "a triangle is isosceles, and its base angles are not equal" is false.

The meaning of II remains unchanged if it is expressed in the following way:

Theorem III. The statement "The base angles of a triangle are not equal, and the triangle is isosceles" is false.

It should be noticed that Theorem I is in hypothetical form, and that II and III are both in conjunctive form. Proposition III differs from II merely in the order of the constituent statements.

Since the conjunctive proposition II is admittedly equivalent to the hypothetical proposition I, the conjunctive proposition III must be equivalent to the following hypothetical form:

Theorem IV. If the base angles of a triangle are not equal, the triangle is not isosceles.

Omitting the illustrative material, the above proof may be reduced to the following:

The Statement

Theorem I′. If h then c

is equivalent to

Theorem II′. The statement "h is true and c is false" is false.

[39] The above symbolism is taken from Hilbert, D. and Ackermann, W., *Grundzüge der theoretischen Logik*, pp. 3–4. Julius Springer. 1928.

But the above has the same meaning as

Theorem III′. The statement "c is false and h is true" is false.

If III′ is now changed to the if-then form, it becomes

Theorem IV′. If c is false, then h is false.

This may be symbolized

$$\bar{c} \rightarrow \bar{h}.$$

But the above proposition is, by definition, the contrapositive of Theorem I.[40]

Since the above proof is general, it is applicable to all theorems irrespective of the number of statements in the hypothesis h, or of the number of statements in the conclusion c. The simplicity of the rule is, however, deceiving; for although it is obvious that the contradictory of $a=b$, is $a \neq b$, it is not so simple to state the contradictory of a conjunct statement like

$$a=b, \text{ and } x=y.$$

Since most of the theorems of geometry contain in their hypotheses more than one statement, it is therefore necessary to find a method for formulating the contradictories of such hypotheses before the rule for obtaining the contrapositive can be applied successfully to all types of theorems.

Contradictory and Contrary Statements

When two persons argue about the color of a certain piece of cloth, one calling it blue and the other green, they are said to be contradicting each other. Again, the belief that "All human beings are selfish" is commonly regarded as the contradictory of the belief that "No human being is selfish." Such uses of the word contradict are not however in accord with the technical meaning of the word. In logic two statements are said to be contradictories

[40] The reader interested in a more rigorous proof should consult any of the following: Couturat, Louis, *The Algebra of Logic*, pp. 26–27. Open Court Publishing Company. 1914. Hilbert, D., and Ackermann, W., *op. cit.*, p. 25, prop. 6. Lewis, C. I., *A Survey of Symbolic Logic*, p. 124, prop. 3.1. University of California Press. 1918. Lewis, C. I. and Langford, C. H., *Symbolic Logic*, p. 34, prop. 3.8. The Century Company. 1932.

of one another, when both cannot be true, and both cannot be false, or in other words, when one must be true and the other false.[41] That the term contradiction cannot be used in this sense in the case of the argument about the color of the cloth is evident from the fact that if the cloth is yellow, both disputants are wrong. Nor does the term "contradictories" apply to the second pair of propositions for both propositions are false if it is true that "Some human beings are selfish" and that "Some human beings are not selfish."

The special designation "contraries" is generally applied by logicians to such pairs of propositions as those quoted above—they cannot both be true, but they can both be false.

The contradictory of the statement, "This piece of cloth is blue" is "This piece of cloth is not blue." It should be observed that one of these statements must be true, and the other must be false.

Likewise, the contradictory of the statement, "All human beings are selfish," is "Some human beings are not selfish," and the contradictory of "No human being is selfish" is "Some human beings are selfish."

Mathematical examples of contradictory and contrary statements. If a and b are two lines in the same plane, then the statement that *a is parallel to b* is contradicted by the statement that *a is not parallel to b*. For, at least one of the two statements must be true, while both cannot be true.

A similar relation exists between the two statements, "m is perpendicular to n," and "m is not perpendicular to n." By like reasoning $A=B$, contradicts $A\neq B$. But note that $A=B$, is not the contradictory of $A>B$, but its contrary. For, if it is the case that $A<B$, both statements are false.

Two contradictory statements must not only exclude each other but must exhaust all possibilities as well. It is of special importance to note the following relations:

$A\leqq B$ is the contradictory of $A>B$
$A\gtrless B$ is the contradictory of $A=B$
$A\geqq B$ is the contradictory of $A<B$

and conversely.

$A>B$ is the contradictory of $A\leqq B$
$A=B$ is the contradictory of $A\lessgtr B$
$A<B$ is the contradictory of $A\geqq B$

De Morgan's Theorem. Most theorems in mathematics contain in their hypotheses more than one statement. The method of contradicting such compound statements is not as simple as it seems. Take for example the conjunctive statement that

I. $a=b$ and angle $x=$angle y.

The contradictory of that statement might seem, on first thought, to be

II. $a\neq b$, and angle $x\neq$angle y.

On second thought, it will be seen that for two statements to contradict each other, it is not enough that they should mutually exclude each other; it is also necessary that one of the two must be true and one must be false. That that is not the relation between statements I and II can be shown in the following way: if it is the case that

$a\neq b$ and angle $x=$angle y,

or

$a=b$, and angle $x\neq$angle y

then both statements I and II are false. Since I and II cannot both be true and both may be false, they are by definition contraries of each other, but not contradictories.

The contradictory of statement I must be, by definition, another statement that will be false, if statement I is true, and must together with statement I exhaust all possibilities. Since statement I asserts that both conjuncts are true, i.e., $a=b$ *and angle $x=$angle y*, its denial must consist of the assertion that *not both conjuncts are true*, or what is the same thing, that *at least one of them is false*.

[41] Cohen, M. R. and Nagel, E., *An Introduction to Logic and Scientific Method*, p. 53. Harcourt, Brace and Co., 1934.
Stebbing, *op. cit.*, p. 58.

In the preceding section it was pointed out that the denial of both statements constitutes a contrary of I but not its contradictory. It will now be shown that the statement,

> III. At least one of the two statements—$a=b$ and angle x = angle y—is false,

may be considered the contradictory of statement I. If the statement $a=b$ is symbolized by p, and *angle* x = *angle* y by q, the following are the only possible relations between them:

1. p is true and q is true.
2. p is true and q is false.
3. p is false and q is true.
4. p is false and q is false.

Statement III comprises within it statements 2, 3, and 4. Since 1, 2, 3, and 4 exhaust all possibilities, their equivalents I and III also exhaust them; since, moreover, I and III are mutually exclusive, they are, then, by definition contradictories of each other. But since statement 1 above and statement I in the preceding column are identical, therefore I and III are contradictories of each other.

The above fact was first pointed out by De Morgan, and constitutes a part of a theorem named after him.[42]

By an analysis similar to the above, it can be shown that the contradictory of a statement of the type

> p is true, q is true, and r is true,

is

> at least one of the three statements—p, q, and r—is false.

With the help of De Morgan's theorem the contrapositive of any theorem can now be formulated, irrespective of the number of data in the hypothesis or of the number of statements in the conclusion.

[42] For a more complete and more rigorous exposition of De Morgan's Theorem, see, Couturat, Louis, *op. cit.*, pp. 32–33. Cohen, M. R., and Nagel, E., *op. cit.*, pp. 68–71, 125. Lewis, C. I., and Langford, C. H., *op. cit.*, pp. 32–33.

Contrapositives of Various Types of Theorems

1. The contrapositive of the simplest type

$$h \rightarrow c$$

follows from an immediate application of the rule to contradict the hypothesis and the conclusion and to interchange their places. It is, of course,

$$\bar{c} \rightarrow \bar{h},$$

> if the conclusion is false, the hypothesis is false.

2. The contrapositive of theorems of the type

$$h_1h_2 \rightarrow c$$

will be, by an application of the same rule,

> if c is false then (h_1h_2) is false.

But since by De Morgan's theorem the denial of the conjunctive statement (h_1h_2) is "at least one of the two hypotheses is false," the expanded contrapositive will then become,

> if the conclusion is false, at least one of the two hypotheses is false.

3. Similarly, the contrapositive of the theorem

$$h_1h_2h_3 \rightarrow c$$

will be,

> if the conclusion is false, at least one of the three hypotheses is false.

4. In general, a theorem of the type

$$h_1h_2h_3 \cdots h_m \rightarrow c_1c_2c_3 \cdots c_n$$

will have for its contrapositive the following:

> If at least one of the conclusions is false, then at least one of the hypotheses is false.

Inapplicability of the Above Contrapositives to Mathematics

Neither analysis nor illustration is necessary to show that although the above contrapositives 2, 3, and 4 are true, they are not the types of theorems one encounters generally in mathematics. It would seem, then, that the promise held out by the contrapositive transformation

is limited to those relatively simple and scarce theorems that have but one hypothesis and but one conclusion. Indeed this apparent impasse may be the explanation for the almost total neglect with which the contrapositive has been treated in the textbooks of Great Britain and of the United States, despite a very auspicious beginning in the last quarter of the nineteenth century.

Fortunately, it is possible to redefine the concept of contraposition and to revive its usefulness by the application of a law of logic that is true only in the case of theorems that have one conclusion, no restriction being made on the number of data in the hypothesis. The statement of the extended definition and the development of the law will be taken up in the next chapter.

SUMMARY

In this chapter, the meaning of the concept and the law of contraposition has been analyzed; a historical study has been made of the attempts to introduce it into textbooks on geometry; and a theory has been proposed to explain its gradual disappearance from current syllabi and books on mathematics.

(To be Concluded)

Some Problems in Evaluation*

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I. Introduction

The mere mention of the word "test" to a group of boys and girls often stimulates a chorus of groans and sighs, and rare indeed is the teacher who is enthusiastic about the task of preparing and marking examinations. It is not unlikely that many of you have recently expressed a fervent wish that you would never again have to struggle through the rites of a departing school year. But testing—or to use a broader term, evaluation of student progress—is a very important part of instructional activity. In a few weeks you will return to your schools and once again face the problems associated with arriving at judgments concerning the educational status of your pupils. It seems to me that the present occasion, before you become subject to the stress and strain of classroom teaching, may be an unusually appropriate time to think about some of these problems. Since at this time we are not faced with the task of planning specific tests for immediate use, we can concentrate upon certain general problems encountered in any careful evaluation of pupil progress.

As individuals we should all be interested in perfecting our testing methods. But we must not overlook the possible effects of such study upon mathematical education as a whole. Most of us have heard some of the questions concerning the educational value of mathematics which are now being so frequently asked. In attempting to answer these questions we describe certain benefits which are supposed to be derived from the study of this subject. We are prone to say, for example, that we are teaching boys and girls to think or to reason logically. Now nearly everyone agrees that this is a desirable educational aim. Moreover, it can hardly be denied that proper instruction in mathematics develops the ability to think with certain types of material. But unfortunately many people doubt that the study of mathematics, other than simple arithmetic, helps much in the effort to cope with most of the problems of day-by-living. We can hardly expect these people to help us accumulate evidence on this question; the task is ours. This means that evaluation programs must include ways of discovering whether pupils have really grown in the ability to solve real problems. We must learn ways of measuring the extent to which pupils are achieving many objectives now often classed as "intangibles." As teachers we have here a problem of our own which is vital and challenging. In my opinion, it is one of the most important of the problems now facing mathematics teachers. Our ability or inability to find a satisfactory solution will be one of the crucial factors affecting the future of mathematics in the secondary schools.

II. Outlines of a General Theory of Evaluation

In solving a problem of this magnitude a general theory is often helpful, and I propose today to discuss the outlines of such a theory. Most of us are so experienced in test construction that it has become almost a routine task. We may not even be conscious that evaluation involves at least four fundamental steps, namely

1. Deciding upon the objectives with respect to which achievement is to be measured,

* An address at the Mathematics Conference, Teachers College, Columbia University, delivered on August 5, 1937.

2. Finding situations in which the student will have a chance to show what he can do,
3. Getting a record of what the student does when faced with the situation, and
4. Interpreting the record.

The teacher of algebra who decides it is about time to find out whether the pupils can solve simple equations has already entered upon the first of these four steps. When that teacher has located or prepared a few simple equations for the class to solve, he has completed the second step. When the class has worked on the problems and handed in the papers, the teacher has thereby obtained a record of pupil performance. Finally, when the papers have been marked and the results studied, the teacher has at least formally completed the fourth step. In such a simple every-day job as this any reference to a general theory seems not only superfluous but actually ridiculous. This is true mainly because the objective to be measured is fairly well understood by the teacher. Moreover, the exercises presented to the pupils are likely to be abstract and precisely similar to those drilled upon in the course of the instruction. Finally, accurate marking of the papers is, in this case, relatively easy. But if this same teacher should happen to decide that it is about time to find out whether the pupils can reason logically in life situations, or whether they really understand the function concept, the problem is not so simple. As one approaches the problem of measuring achievement of objectives which have hitherto been regarded as intangible, the recognition of the four steps outlined above is of great value. Without such an outline a brief exploratory foray into the wilderness surrounding the goal may be followed by a humiliating retreat. But with such an outline in mind the problem is divided into parts which, while still difficult, may be successively attacked and solved. The remainder of this discussion will outline some of the difficulties encountered in such a project, and will attempt to give some helpful suggestions for overcoming them.

III. The Formulation of Objectives

The first step in any evaluation program is to decide upon the objectives with respect to which achievement is to be measured. Let us therefore consider briefly some methods of obtaining lists of objectives. One such method consists of making an analysis of the purpose of the course—that is, of trying to formulate answers to such questions as "Why should algebra be taught in the secondary school?" A second method frequently used consists of analyzing textbooks on the subject and attempting to identify the purpose of each topic. In practice a combination of these two methods is highly desirable.

In this connection I would like to make a remark which I think has some bearing on current issues in mathematical education. When a research mathematician has been persuaded to address a group of secondary school teachers he not infrequently touches upon the values of the study of mathematics. His broad understanding of the field enables him to describe values or objectives in rather general terms. These papers, then, are one invaluable source of authoritative statements on the broad purposes of mathematical instruction. Unfortunately, however, when one examines the elementary textbooks in the subject one sometimes finds it extremely difficult to make the connection between these broad statements of value and the specific tasks set for the pupils. It seems to me we must try to do two things: first, make such connections fairly clear to competent critics, or eliminate the topic in question; and second, as I said before, we must be able to show that our instruction is really achieving the accepted purposes. This involves analyses of the types mentioned above. If our objectives deal only with the

specific facts and mechanical skills of mathematics perhaps such analyses are unnecessary. Most of us know what these facts and skills are, and we can go about the job of teaching the pupils the maximum number possible in the time available. But if there are broader objectives (and we all believe there are) we must identify them and show we are achieving them before the time now placed at our disposal vanishes.

Now what are some of the difficulties one meets in attempting to formulate a satisfactory list of objectives? One difficulty is that objectives obtained from the first type of analysis are often stated so vaguely that teachers are not sure what is meant. The interpretations which are given to such statements depend upon the mathematical and educational training of the interpreter, and are thus subject to rather wide variation. If teachers do not clearly understand the implications of such statements, it is not likely that they are actively working upon such objectives in the classroom. Moreover, methods and materials for fostering achievement of these values are, for the most part, either not well known or are entirely lacking. This situation has led some writers to classify objectives as "ultimate" and "immediate," and for practical reasons to concentrate upon the measurement of achievement of the specific skills and concepts commonly recognized as of the latter type. This leaves still to be solved the important problem of showing that the "ultimate" objectives are being achieved.

The second method of obtaining objectives involves difficulties of a different sort. Instead of being too vague, the statements now tend to be too specific and too numerous. They are usually stated in terms of specific content; for example, "the ability to multiply signed numbers," or "the ability to square binomials." When objectives are stated in this way, the list tends to become so long that it is difficult to keep all of the items in mind. Points of major emphasis are often lost sight of because of the extensiveness of the list. By mentioning specific content, statements of this kind tend to restrict rigorously the curriculum to the items mentioned in the prescribed list. Teachers feel there is little opportunity for experimental efforts to develop broader aims which call for methods and materials other than the drill type.

In order to overcome these difficulties we must seek a form of statement of objectives which somehow strikes a happy medium between vagueness and over-specificity, and such that the list is reasonably complete, yet not too extensive. One criterion which helps realize these purposes is the following: The objectives should be stated in terms of the behavior patterns expected of pupils who make progress toward achieving the objective. The term "behavior" is here used in the broad sense which includes mental and emotional reactions as well as those observable in terms of overt action. The criterion means that we must try to describe what pupils do, think, or feel, when they have achieved the objective. The effort to formulate such statements usually results in clarification of the meaning of the objective, and it points the way to improved methods of teaching and evaluation.

For purposes of illustration, let us consider one of the objectives now usually classified as "ultimate" or "intangible"—for example, "appreciation of beauty in the geometrical forms of nature, art, and industry." We must attempt to tell what a person does when he appreciates—in a sense, to define the word in operational terms. Among the kinds of behavior which might be listed are the following:

1. He recognizes geometrical form in natural objects (such as crystals), in artistic products (such as stained glass windows and architectural design), and in industry (such as the Texaco and Chevrolet trade-mark designs, machine parts, etc.).

2. If occasion demands it, he is able to defend preferences in terms of certain principles of design such as symmetry and proportion.
3. He calls the attention of his friends to examples of the use or occurrence of geometric form. Occasionally, he may bring examples to class for the benefit of classmates and teacher. Thus, a girl on a summer European trip took time to sketch the design of a certain cathedral window and brought the drawing to class in the fall.

This list could be extended, but perhaps enough has been said to suggest what is meant by description of the objective in terms of behaviors.

IV. Choice of Situations

The second step in the general outline of procedure is to describe some typical situations in which the student will have an opportunity to exhibit one or more of the behaviors listed. It is at this point that one's attitude toward transfer of training becomes important. If the teacher is interested in transfer, then some of the situations should be bona fide life-situations. Consider "the ability to reason logically." If by this we mean only that the pupil is to be able to give acceptable proofs of geometrical exercises, then perhaps the teaching and testing need not go beyond the confines of pure mathematics. But if we mean that the pupil should also be able to reason logically as he reads newspapers and magazines, then the available evidence tends to show we must modify our teaching and testing materials accordingly. A student who apparently understands the "if-then" type of argument in geometry may fail to reach the logical conclusion in a situation involving social or political issues. If we want to find out whether he can make the transfer, we must study his reactions in situations which require it. Such situations can be found, although the teacher who is unaccustomed to looking for them often finds it difficult at first. This difficulty can be overcome with practice—that is, as the teacher himself learns to make the necessary transfer.

V. Obtaining a Record

The third step in the general procedure consists in obtaining a record of the student's behavior when faced with an appropriate situation. In some cases, this may involve observation and recording of the observed behavior. Suppose, for example, the objective is that the student should learn to handle his compasses, or perhaps a sextant, skillfully. We can then make it a point to watch him as he attempts to draw circles or measure angles with the sextant and make notes of his progress as he becomes more skillful in the use of the instrument. We thus see that it is not necessary to assume that evaluation involves paper and pencil tests. There are several other useful methods of getting a record of behavior. Of course, paper and pencil tests are an economical method to use if the objective is such that they can be used. The pupil then makes his own record. But as we attempt a comprehensive evaluation program which includes "intangible" objectives, the notion that our aim at this stage is to obtain a record is extremely valuable. Many people have given up in despair at this point because they have assumed at the outset that a paper and pencil test marked by the student is the only method of obtaining a record of achievement.

Fortunately, many objectives lend themselves to paper and pencil tests. But here again much progress has been blocked by assuming that some form of short-answer technique should be used. The marking of essay-type tests is notoriously subjective. The same paper evaluated by different teachers may receive a wide range of scores. Essay examinations also require so much time to be spent in the mechanics of writing they they ordinarily sample only a small range of pupil be-

havior and are consequently not very reliable. To overcome these difficulties various short-answer techniques have been devised. Thus we have true-false tests, multiple-choice tests, matching tests, and the like. Ordinarily the use of these techniques reduces subjectivity in the scoring, but it involves other difficulties. Many teachers reject tests of this type because they feel that the tests do not really measure achievement of the objectives for which they are designed. The true-false tests in particular have often been subjected to severe criticism. Many teachers prefer examinations in which pupils write out complete proofs or show all of the work necessary to solve a problem. Advocates of one or the other of these two types of tests thus appear to be in opposition to each other. It is commonly assumed that this is an either-or situation. Some teachers thus reason as follows: Either we must continue to depend upon tests calling for reasonably complete solutions or proofs, or we must resort to short-answer techniques. Since we are not convinced that techniques such as the use of multiple-choice items give us satisfactory evidence of achievement, we must continue to use the traditional form of examination.

Fortunately, there is a way out of this difficulty, but it involves considerable work. For purposes of illustration, let us consider the problem of measuring the ability to prove originals in geometry. Suppose we prepare a test calling for the proof of a number of originals and give it to a group of pupils. If several teachers mark the papers considerable variation in the scores may be expected. This is because the several teachers are scoring on different bases. Analysis of the objectives which underlie the scoring will bring these differences to the surface. It is then usually possible to prepare a set of scoring directions which depend quite definitely upon the objectives really deemed important. If the papers are now re-scored according to the directions, it is ordinarily possible to get rather close agreement between the scores assigned by the different teachers. This same method may be applied to any good essay-type test, even when the objectives covered are decidedly of the "intangible" sort. In other words, it is possible to eliminate most of the subjectivity of scoring provided enough pains are taken in preparing a key and marking the papers. The scores thus obtained may be regarded as the *basic evidence* provided the teachers agree that this type of examination calls for abilities they regard as important.

Suppose now we prepare a short-answer test covering exactly the same situations —for example, exactly the same originals. If we give both the longer essay-type test and the test using the new techniques to the same group of pupils, we can compare the results obtained from the two types of tests. If the results agree very closely, then it is possible in the future to use the shorter form which requires less writing by the pupil and is much easier to score. In general, then, the aim is to devise some short-answer form of test such that the coefficient of correlation between the results on this test and the original basic evidence is high—say, about .90 or higher. We do not claim that the behavior response is the same in each case. But one set of scores may serve as an index of the other. Such a process as this requires more detailed work than the average teacher is prepared to do. But once we learn how to make newer types of tests which give valid evidence of achievement of important objectives it is often possible to short-circuit much of the work.

In order to obtain a sufficiently high coefficient of correlation the new-type test may appear very much different from the types with which most teachers are familiar. In order to test some of the objectives hitherto regarded as intangible, we must expect test situations which are somewhat more elaborate than those now given. The tests will appear to be more complex than most published tests. But

it seems reasonable to assume that in order to measure complex behavior a more complex instrument is needed. It should be possible, however, to devise means of obtaining evidence about achievement of objectives now considered intangible, and study of this evidence should lead to marked changes in our teaching procedure. The point at the moment is that in the approach outlined above we need not begin by assuming that any of the well-known short-answer techniques are to be used. We first get some direct evidence—by actual observation if necessary. We then try to find an easier way of getting essentially the same evidence. If a short-answer technique is satisfactory, we can use it. If not, no claim need be made for it. But we are then forced back upon the original or basic evidence.

VI. Interpreting the Results

The fourth step in general testing procedure is the interpretation of the results. Unfortunately the chief difficulty here is the notion that the main purpose of testing is to assign marks to the pupils. We should be more concerned about the *growth* of students with respect to the objectives of the course. This means that we need a variety of tests aimed at the different objectives of the course. We should measure the students at the beginning and periodically thereafter—if possible, even several years later. We should then report to parents, administrators, and higher institutions in terms of the growth (or lack of growth) and powers of retention of the pupil with respect to all of the really important objectives of instruction. Our problem is to construct a picture of the pupil which reveals his achievement in terms which are readily understandable to the layman. It is doubtful if this can be done by merely adding together test scores and assigning a numerical or letter grade.

Perhaps an illustration will help to make clear certain broader types of interpretation of pupil behavior. Consider the following exercise taken from a test calling for certain abilities in interpreting data.

Directions: *In each of the following exercises some test, experiment, or situation is described. Below the description you will find several statements which are suggested as possible interpretations of the data. Assume that the facts of the description are accurate. Carefully consider each of the statements and indicate in the columns to the right whether you believe*

(1) the evidence is sufficient to make the statement true.
(2) the evidence is sufficient to make the statement false.
(3) the evidence suggests that the statement is probably true.
(4) the evidence suggests that the statement is probably false.
(5) the evidence is insufficient to make a decision concerning the statement.

Problem 5:

The surface temperature, distance from the sun, and the diameters of five planets are listed in the following table:

Planet	*Average Temperature (absolute)*	*Distance from the sun (millions of miles)*	*Diameter*
Mercury	650	36	3009 miles
Earth	290	93	7918 "
Mars	250	141	4339 "
Jupiter	100	483	88392 "
Saturn	90	886	74163 "

Statements	(1)	(2)	(3)	(4)	(5)
a. The amount of heat falling on planets close to the sun is greater than that falling on planets farther away. .					
b. The temperature of the planets becomes less as their distance from the sun increases.					
c. The temperature of a planet depends solely upon the radiant energy which it receives from the sun.					
d. The temperature of the planets becomes less with increasing diameters.					
e. Planets farthest from the sun have lower temperatures than those nearby.					

Statements	(1)	(2)	(3)	(4)	(5)
f. Uranus, a planet at a distance of 1781 million miles from the sun, will have a lower surface temperature than Saturn.					
g. The distance from Mars to the Earth is 48 million miles.					
h. The temperature of Venus, at a distance from the sun of 67 million miles, is higher than that of Mercury. .					
i. Jupiter is the largest planet of the solar system.					

After the statements have been prepared, a scoring key is agreed upon by a jury of competent teachers. Let us observe first that item *g* is of the type commonly found on true-false mathematics tests. An easy computation shows that on the basis of these data it is true. All of the other statements are subject to certain qualifications, and if the test offered only the choice between "true" and "false" careful thinkers would be uncertain how they should be marked. The technique used here gives them a chance to express that uncertainty. Poor thinkers, on the other hand, tend to be more sure of themselves, and, failing to see the qualifications, are quite satisfied to call such statements absolutely true or false. Good thinkers recognize that the data here given refer to only five planets, but all of the statements except *g* depend in one way or another upon whether the facts about these five actually disclose the real trend. Is it safe to generalize from a few statements? In the case of statement *d*, we see definitely that it is not, and this statement can be marked false. Statements *b* and *e* are true in the case of the five planets given, but since the generalization may break down, careful thinkers mark them as only probably true. Over-hasty generalizers consider them true. Statement *a* is still more doubtful since it refers to *amount of heat* while the data refer to surface temperature. Statement *f* requires an extrapolation, and statement *h* requires an interpolation. If the trend holds, the first is true and the second false, but careful thinkers consider them only probably true and probably false on the basis of these data only. Good thinkers consider statements *c* and *i* entirely uncertain on the basis of the data given. As a matter of fact, the first is probably false, and the second probably true, but these data do not establish that. It is worth noting that these statements call for some ability to recognize relationships and some insight concerning the nature of inductive thinking, interpolation, and extrapolation.

A test made up of a number of problems such as this and using different types of data clearly demands more thinking than is required by the usual simpler type of test situation. The student may respond to a given statement in five different ways. But the technique permits much more significant interpretations than the usual five-choice type of item. Analysis reveals that some students tend rather consistently to say that items which are considered true by the jury are only probably true, or are uncertain. Such students tend to be cautious or even over-cautious relative to the jury. Other students tend to mark *uncertain* or *probably true* statements as probably true or true, respectively, or even to mark *uncertain* statements as true. Such students tend to be "gullible." Others agree closely with the key and are considered good interpreters. Still others make wide errors in judgment. Interpretations of these and other sorts can easily be made with the help of summary sheets conveniently arranged for the purpose. It is even possible to score and analyze hundreds of test papers of this type in an hour with the help of an electrical test-scoring machine. More important, however, than numerical test scores is the evidence of behavior tendencies like gullibility or cautiousness. A comprehensive verbal picture of an in-

dividual student involves the synthesis of many interpretations of this sort, based on tests for a variety of different objectives. When we are able to supply interested parties with such data in place of the customary "87% in algebra" type of report, we shall have made great progress toward accurate description of the development of pupils.

VII. Conclusion

This general discussion of evaluation problems will have served its purpose if it has in any way broadened your point of view. Progress in testing depends upon teachers in the field more than upon any other single factor. Only when teachers believe that it is possible to measure achievement with respect to the real objectives of mathematical instruction will they make serious efforts to do so. Many will only believe this when they have been shown how it may be done. The theory which has been discussed today is yielding promising results in practice. If it or a better theory becomes actively and widely applied by teachers in the field, we can hope to establish the values of instruction in mathematics upon a sounder basis of evidence than we now have.

◆ THE ART OF TEACHING ◆

The Concept of Approximation

By Edyth Clarke Alexander
Eastern High School, Washington, D.C.

An important concept of mathematics which relatively has been overlooked in the mathematics courses of study in our American schools is the concept of approximation.

Measurement is a complicated process, the precision of which depends upon many factors. The kind of measuring rod used, together with its imperfections such as warping etc. enter into the precision of the measurement. The position (directly over or slightly to the right or left) of the one reading the measurement influences the numerical expression of the performed measurements. Shifting the measuring rod and the difficulty of keeping it in a straight or horizontal line also cause slight deviations. The personal equation of eyesight and judgment of the reader enter into the measurement.

The history of the origin of measurement is of interest to secondary school students. They will be interested to learn that a Cubit was the length of a forearm from the end of the middle finger to the point of the elbow—about 20 inches; the digit about .72 to .75 inch was the breadth of a finger and that it was decreed that three barley corns placed end to end equaled an inch. A brief history of these first linear standards from parts of the human body to the more recent development of measuring instruments like the Vernier Caliper and Metric Gage Blocks will introduce the pupils to an enthusiastic study of measurement.

The teacher should introduce into the classroom weighing scales, ruled tapes, protractors, clocks, etc. and the pupils by making repeated measurements will see for themselves the approximate nature of measurement. Rounding off numbers will take on a new significance to pupils who have had this type of preliminary experience.

The gathering and organizing of data may seem to fall outside the domain of a mathematics course and belong only to laboratory courses, yet it is a valuable and necessary experience for all pupils to have and should be introduced into our mathematics field.

Pupils always are interested in their results cn tests and this affords the teacher an excellent opportunity to depart from the ordinary procedure of scoring the test on the 100% basis and introduce median, mode, standard deviation, and other statistical terms thereby teaching the ideas of central tendency, variability and dispersion.

The slide rule affords excellent opportunity for the development of the notion of arithmetic mean, geometric mean, logarithms, and the idea of limited accuracy.

This concept of approximation develops common sense and leads away from formal technique. It correlates with other subjects and makes the pupil appreciate the services and usefulness of mathematics, yet in no way detracts from one of mathematics' most conspicuous virtues—namely exactness. The pupil will grasp the fact that counting is exact but other measurement is approximate. While we are teaching the concepts of solution, proof, relationships and operation, let us not neglect in our mathematics courses this important concept of approximation.

A Letter from the New President of the National Council of Teachers of Mathematics

TEACHERS OF MATHEMATICS:

The presidency of the National Council of Teachers of Mathematics is an honor and a responsibility of no mean significance. I greatly appreciate the honor of being president of an organization which for nearly twenty years has fought valiantly and successfully for the improvement of mathematics teaching.

For several years one annual meeting, the organization of local affiliated clubs, and the publication of *The Mathematics Teacher* were the only activities through which the influence of a few heroic pioneers was felt. Then yearbooks began to appear. We have now published twelve yearbooks, many of which have been really outstanding contributions to education. The influence of the Arithmetic Yearbook alone has been unpredictably great. Finally the annual meetings have been extended to three per year, one with the School Administrators in February, the second with the classroom teachers at the National Education Association in June, and the third with the Mathematical Association in December. Thus the National Council has expanded until it now presents a varied program with immeasurable possibilities for influence.

It would be impossible to give due credit for the achievements of the past years. Great pioneer personalities have guided our destiny and dominated our actions. We are grateful for these strong minds with high ideals and I trust that some competent person will pay a deserved tribute to each one. The Council owes a conspicuous and unpayable debt to the tireless energy, the refreshing ideas, and the genial influence of the present editor of the magazine and yearbooks. Dr. Reeve's unselfish devotion and boundless enthusiasm coupled with his financial sagacity have been largely responsible for the high ideals of the Council and for its increasingly competent financial structure. To the associate editors, to the members of the Boards of Directors, to the other officers, and to former presidents we owe much for their "everlasting team work" which has made the Council succeed. The deep interest and intelligent enthusiasm of Martha Hildebrandt have been largely responsible for the continued growth and influence of the council during the last two years. As president she has shown marvelous diplomacy and keen insight. She has guided the Council during trying years to new heights of influence and success.

My purpose in writing to you, mathematics teachers, is neither exposition nor eulogy, desirable and proper though they may be. It is rather to call your attention to the possibilities for the future. Our educational system is in a state of flux. Sanity does not always control the reformer who feels that change is progress, who would toss the motor out of his car because the spark plugs were fouled, who would cast aside all old for anything new. In a world growing increasingly complex, becoming constantly more scientific, and demanding more and more accurate thinking the language of quantity will become increasingly necessary for successful social participation. Mathematics teachers who are interested in substantial education must unite to improve the teaching of mathematics, to delete the useless, and to make the useful functional and meaningful. It is only by concerted action that we may get conspicuous and significant progress and attainment.

As your newly elected president of the National Council of Teachers of Mathematics I invite your cooperation. If you have suggestions for more effective yearbooks, meetings, magazine articles, or other activities, think them through carefully and send them to me. If you have ability, do not "hide it under a bushel," let it out. The National Council is an ideal testing ground. At its meetings you will find genial companions, stimulating pioneer spirits, and courageous endeavor. Let us pull together for a bigger and better Council with more effective influence for insuring to every child a really substantial education.

Respectfully yours,

H. C. CHRISTOFFERSON

EDITORIALS

The National Council's New President

Dr. H. C. Christofferson of Miami University at Oxford, Ohio has taken up his duties as President of the National Council of Teachers of Mathematics. His experience as presiding officer at several past meetings of the Council will not only be helpful to him in his new role but it has given Council members a chance to find out what a good presiding office he would make.

Dr. Christofferson received his B.A. degree at the University of Minnesota in 1917. After six year's experience in the Minnesota schools as a high school teacher, principal and school superintendent, he received the M.A. degree at the University of Chicago in 1923. He was then head of the Department of Mathematics at the State Teachers College at Oshkosh, Wisconsin. Since 1928 he has been Professor of Mathematics at Miami University and since 1935 he has also been Director of Secondary Education at Miami. He was awarded the Ph.D. degree by Columbia University in 1933.

Dr. Christofferson is well fitted by experience, training, and personality to serve the Council in many useful and diverse ways during his term of office. *The Mathematics Teacher* wishes to take this opportunity to extend to him its best wishes for a happy and successful term and to congratulate the Council upon his elevation to the presidency. W.D.R.

Renewal of Membership

During the trying time that we have gone through this year due to the untimely death of Miss Mabel Winspear, the genial secretary for *The Mathematics Teacher*, we have tried to keep many subscribers on our list often months after their subscriptions had lapsed in order that they might not miss any issues. In some cases we inadvertently removed a few subscribers from the list who should have been kept there. In all such cases we have tried to straighten out such matters to the best interest of all concerned and we shall feel sorry to learn that anyone has been deprived of the magazine who really wished to receive it. It is obvious that we cannot any longer continue the practice of keeping people on our mailing list who do not pay their subscriptions promptly as such failure to pay causes us no end of trouble and unnecessary labor and expense. In the future we shall, therefore, be compelled to drop members from the list who upon notification that their subscription has expired, do not then remit $2 for the ensuing year. This applies particularly to those whose subscriptions expire in May 1938 and for whom there may be temptation to put off payment until after the summer vacation. You will help us greatly and will save yourselves possible trouble later if you will send in your renewals promptly.

W.D.R.

IN OTHER PERIODICALS

By NATHAN LAZAR
Alexander Hamilton High School, Brooklyn, New York

1. "The Bearing of Higher Geometry on the School Course." *The Mathematical Gazette.* 21:338–359. November, 1937.

A report of the proceedings of a meeting of Section A of the British Association that took place at Nottingham, on September 7, 1937.

The following is a list of the participants in the symposium and of the topics they discussed.

a. Piaggio, H.T.H., Introductory Remarks.
b. Neville, E. H., "The Influence of the University on School Geometry."
c. Green, H. Gwynedd, "Infinity in Euclidean Geometry."
d. McCrea, W. H., "The Circular Points and Elementary Geometry."
e. Ruse, H. S., "Differential Geometry."

2. Boyd, Rutherford, "Mathematical Themes in Design." *Scripta Mathematica.* 5: facing pages 5 and 44. January, 1938.

Two beautiful reproductions of "still" pictures used in the mathematical film "Parabola." Apropos this film, the following note appears on page 71 of the same issue of *Scripta Mathematica:* " 'Parabola' is a new film in which mathematics and art are linked together; it was designed and directed by Rutherford Boyd whose mathematical designs published in *Scripta Mathematica* and other magazines have appealed to both artist and mathematician. The photography is by Ted Nemeth. . . . This unusual cinema conception will be first exhibited in New York early in January. Additional information regarding the film may be obtained by writing to Rutherford Boyd, 112 Prospect Street, Leonia, New Jersey."

3. Copeland, Harold W., "Modernizing Mathematics," *Bulletin of the Kansas Association of Teachers of Mathematics.* Vol. 12, No. 2, pp. 3–5. December 1937.

The author believes that a modern mathematics program should have the following characteristics:

1. "It should be taught on a scale to understand better community affairs. . . .
2. "Introduce and keep the student keenly aware of logic and its value to the solutions of problems in everyday life. . . .
3. "Such a program of teaching mathematics should introduce a means of worthy use of leisure time.
4. "No modern mathematics program would be complete unless it attempts to develop those powers of understanding and of analyzing relations of quantity and space which are necessary to an insight into our invironment."

4. Georges, J. S., "The Operator 'J' and Errors in the Learning of Mathematical Concepts and Processes." *School Science and Mathematics.* 38: 143–145, February, 1938.

The theme of the paper "is a brief study of the properties of mathematical relationships such as the reflexive property, the transitive property, etc. An abstract symbol is used to represent different concepts and processes in mathematics. In terms of this symbol properties of such concepts and processes are analyzed."

An interesting application of the above analysis is made to the explanation of certain type errors made in arithmetic and algebra.

5. Lloyd, D. B., "Bibliography of Popular Mathematics." *School Science and Mathematics.* 38: 186–193, February, 1938.

The experiences the author had as a faculty adviser of mathematics clubs led him "to the belief that there exists a need for a broad, usable bibliography of popularly written material on mathematical subjects—a storehouse that can be drawn upon in accordance with the needs and interests of those in the club."

The bibliography is divided into two sections: Books and Periodicals.

Under the heading Books, the following subheadings are found: 1. History, 2. Biography, 3. Recreation, 4. Enrichment and Source Material.

Under the heading Periodicals, the following 16 headings appear: 1. Mathematics—Appreciation. 2. Mathematics—Relation to other fields. 3. Mathematics—History. 4. Mathematicians. 5. Arithmetic. 6. Numbers. 7. Algebra. 8. Geometry. 9. Trigonometry. 10. Probability. 11. Measurement, time, etc. 12. Physics, astronomy—fourth dimension. 13. Recreation and problems. 14. Devices, equipment. 15. Programs and Contests. 16. Plays.

About 50 books and 200 magazine articles are referred to in the bibliography.

6. Morris, Richard, "Mathematical Induction for Freshmen." *National Mathematics Magazine.* 12: 183–187, January, 1938.

After pointing out the various difficulties students usually find in proofs by mathematical induction, the author proceeds to an excellent and detailed presentation of that topic. "Some instructors may contend that this form of treatment is too formal or too prolix. But our contention is that such formality and detailed statement of the steps are the very type of work that a freshman in college needs, not only for his work in mathematics but for his other studies, particularly scientific studies."

7. Swenson, John A., "The Newer Type of Mathematics Compared with the Old." *School Science and Mathematics.* 38: 107–112. February, 1938.

After characterizing briefly the three kinds of mathematics—compartment mathematics, general mathematics, and integrated mathematics—the author proceeds to point out the three ways in which the mathematic curriculum in the secondary schools of the United States is defective: concepts, time distribution and continuity. The newer mathematics which the author advocates not only remedies these defects but also "concerns itself with topics which lead to the mathematics of social science, since science no longer means merely physical science." A short historical section traces the recent changes in mathematics to the recommendations made by the International Congress held in Rome in 1908.

8. Taylor, Katherine, "The Learning Process." *Bulletin of the Kansas Association of Teachers of Mathematics.* Vol. 12, No. 2, p. 3. December, 1937.

"Ideas can be as real to children as are the things they see and handle. . . . The things that often bewilder them in the learning of abstractions is the fact that most of us teachers habitually thrust the label for an idea upon our pupils before they have had a chance to learn what the idea is. . . . If practice is introduced too soon, before the meaning is clear, it often becomes merely a game of chance, and the integrity of learning is endangered."

9. "The Unification of Algebra in Schools." *The Mathematical Gazette.* 21: 314–337. November, 1937.

A report of the proceedings of a meeting of Section A of the British Association that took took place at Nottingham on September 4, 1937.

The following is a list of the participants in the symposium and of the topics they discussed:

1. Broadbent, T. A. A., "An Introductory Survey of the Present Situation."
2. Parsons, G. L., "The Introduction of the Fundamental Ideas of Algebra."
3. Newman, M. H. A., "The Course as Seen from the University."
4. Langford, W. J., "The Teaching of Algebra in the Advanced Forms of Schools."